Optimal Monetary Policy in a Collateralized Economy

Preliminary

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Abstract

There is a convenience yield on U.S. Treasuries. So, the central bank's open market operations exchange one type of outside money (cash) for another type of outside money (Treasuries). When Treasuries are scarce, the private sector produces substitutes, mortgage-backed securities (MBS) prior to the recent crisis. The quality of the collateral in the economy matters. When the ratio of MBS to Treasuries is high, a financial crisis is more likely. In boom times, macroprudential policy aims to keep the MBS-Treasury ratio low, but the central bank also aims to increase the real amount of collateral and hence output. This introduces a tradeoff between expansionary policy and macroprudential policy. We analyze optimal central bank policy in this context as a dynamic game between the central bank and private agents. The equilibrium concept we use considerably simplifies the analysis of this type of game. In equilibrium, the economy will experience deflation during recessions, and a boom-bust pattern is the equilibrium outcome under the optimal policy.

1. **Introduction**

In this paper, we introduce a macroprudential policy for the central bank to pursue to manage financial fragility. We show that this policy interferes with what would otherwise be the optimal monetary policy (in an economy which never has crises), but that this interference is optimal to reduce financial fragility.

The central bank's problem arises in the following way. There is a convenience yield on U.S. Treasuries. So, the central bank's open market operations exchange one type of outside money (cash) for another type of outside money (Treasuries). When Treasuries are scarce, the private sector produces substitutes to use as collateral and as a store of value, mortgage-backed securities (MBS) prior to the recent crisis. When the ratio of privately produced "safe debt" (MBS) to Treasuries is high, i.e., there is a shortage of government-produced safe debt, a financial crisis is more likely because the privately-produced safe debt, used to back short-term bank debt (repo and asset-backed commercial paper) is not riskless. The central bank cares about the quality of collateral in the economy. So, here macroprudential policy aims to keep the MBS-Treasury ratio low to reduce financial fragility. This introduces a conflict between maximizing output and minimizing financial fragility.

We analyze optimal central bank policy as a dynamic game between the central bank and private agents. The equilibrium concept we use considerably simplifies the analysis of this type of game. In equilibrium, the economy will experience deflation during recessions, and a boom-bust pattern is the equilibrium outcome under the optimal policy.

We start from the following stylized facts:

1. There is a convenience yield associated with U.S. Treasury debt. See, e.g., Duffee (1996), Krishnamurthy and Vissing-Jorgensen (2012, 2015), Nagel (2014) and Gorton and Muir (2015).

2. When Treasury debt is scarce, the convenience yield rises and the private sector produces substitutes, mortgage-backed securities in the recent crisis. See Krishnamurthy and Vissing-Jorgensen (2015), Gorton, Lewellen and Metrick (2012), Xie (2012), and Sunderam (2012).

3. Credit booms—high growth in private debt—typically precede financial crises. See, e.g., Schularick and Taylor (2012), Jorda, Schularick and Taylor (2011), Laevan and Valencia (2012), Desmirguc-Kunt and Detragiache (1998) and Gorton and Ordoñez (2015).[1]

4. When the ratio of Treasury debt to GDP is low, a financial crisis is more likely. Alternatively, when the ratio of MBS to Treasuries is high, crisis is more likely. This is implied by points (2) and (3).

[1] The literature on credit booms and crises is large and we have only cited a few of the many papers.

“Macroprudential policy” here is taken to mean optimal management of the quality of the collateral in the economy, the ratio of MBS to Treasuries. Financial fragility is increasing in this ratio, and this reduces welfare. More generally, the ratio would be credit to the private sector divided by Treasuries. MBS and Treasuries are used to back repo, money market funds and mortgage-backed commercial paper, i.e. short-term debt which is the root of crises. Here, rather than regulate the quantity of short-term debt directly, the central bank regulates indirectly, via collateral quality. This is very natural since open market operations already trade cash for Treasuries, and vice versa. So, whether or not the central bank recognizes this, it is in fact affecting the quality of collateral in the economy. Here, the central bank explicitly recognizes this. Macroprudential policy is not distinct from monetary policy.

Since nonfinancial firms are the end-users for Treasuries and MBS, they are one clientele demanding one kind of money, Treasuries. Households are largely the source of the demand for traditional money, M1. From the viewpoint of these clienteles, Treasuries and MBS are not substitutes for cash or demand deposits. Taking account of both types of money has important implications for monetary policy since there are now two clienteles demanding two different types of “money.” Since each clientele has a demand for its type of money, optimal monetary policy has a difficult trade-off in terms of welfare. On the one hand, each clientele needs “money”, but on the other hand inflation is a function of cash.

This change in the composition of money demand, fewer demand deposits and more short-term debt backed by MBS, is evident in the data. Gorton, Lewellen, and Metrick (2010) document that the share of safe debt in the U.S. economy, including both U.S. Treasury debt and privately-created near-riskless debt has remained constant as a percentage of all U.S. total assets since 1952. This is akin to a stable money demand function. But, while the share has remained the same, the composition of the privately-produced money has radically changed, with a significant decline of demand deposits and a significant rise of MBS and money market instruments.

Gorton and Muir (2015) describe this change in the composition of money as corresponding to a transition from a system of immobile collateral – bank loans staying on bank balance sheets, to a system of mobile collateral, which is created by securitizing bank loans. Prior to this transformation, the central bank did not need to include collateral quality in monetary policy because demand deposits were insured and bank examiners monitored bank loans (collateral). The issue we address here arises in a world where collateral is mobile and privately-produced collateral consists of mortgage-backed securities.

We model these two demands for money in the following way. As in Sidrauski (1967) and others, we model the household demand for money by entering money into the household utility function. Firms demand Treasuries and MBS, implicitly to use as collateral to borrow. Financial intermediaries will briefly appear in the model, but the use of Treasuries and MBS to back repo, for example, is implicit. The demand for this second kind of money is modelled by specifying production to be a function of real Treasury securities and MBS, recognizing that privately-created MBS cannot supply as much “liquidity” as Treasuries. The private sector cannot produce perfect substitutes for Treasuries. Entering Treasuries and MBS into the production function is a reduced form for the combined financial and non-financial sector. Nonfinancial firms need collateral to borrow. See, e.g., Gorton and Ordoñez (2014). As we will explain, securitization plays an explicit role in the model. Securitization of private housing-related assets is the endogenous source of MBS in the model.

Expansionary monetary policy causes inflation while generating more private liquidity in the economy. To do this, the central bank needs to buy Treasuries. But, buying Treasuries results in more MBS being endogenously produced. Increasing MBS—a credit boom—increases financial fragility. This is the central bank's policy conundrum. When the stock of MBS in the economy is high, to mitigate financial fragility, the central bank must "take the punch bowl away," by selling Treasuries, i.e., a deflation. In the infinite horizon model, this will introduce a rich set of dynamics.

The model we analyze is an infinitely-repeated game between one large player—the central bank---and many small players, agents in the private economy. It is a Ramsey problem in which the central bank cannot commit to its optimal policy. Such settings have been the subject of a large amount of research because of the results of Kydland and Prescott (1977), showing that dynamic programming cannot be used as a solution method because of the dynamic inconsistency. A recursive characterization of Perfect Public Equilibrium (PPE), which was first defined in Fudenberg, Levine, and Maskin (1994), for a dynamic game was proposed by Abreu, Pearce, and Stacchetti (1986) (APS). APS sheds light on this issue. In any PPE, the strategy of the large player is dynamically consistent although there is no commitment. Moreover, APS, in a game with no large player, show that past histories can be summarized by promised future utilities, continuation values, and the values of agents can be described recursively. This approach has been widely used in macroeconomics.[2]

Our solution method is most closely related to Phelan and Stacchetti (2001) (PS), an extension of APS for a strategic game between a large player and a continuum of small players, where there is a public state variable.[3] PS do not augment payoffs using continuation values directly as APS do, but rather write the continuation values as a product of the choice variables and marginal values of these variables. By augmenting the payoffs in this way, agents are enticed to stay on the equilibrium path, without a need to characterize the payoffs for agents off the equilibrium path. This captures the key feature of small players, that is, as a price-taker, an individual small player's action does not affect the other small players' payoffs.

We closely follow PS with one important difference. The difference concerns the equilibrium concept. PS use Perfect Public Equilibrium and determine the *set* of equilibria by determining the appropriate correspondences. Because functions cannot be used, it is very difficult to

[2] See Ljungqvist and Sargent (2004), chapter 22, for a summary. An alternative, closely related, approach introduces Lagrange multipliers as co-sate variables. See Kydland and Prescott (1980) and Marcet and Marimon (2011).

[3] Atkeson (1991) showed that APS can be used when there is a publicly observed state variable.

analytically solve the game or to compute the equilibrium numerically.[4] Computation of the equilibrium is also an issue in closely-related models, such as Chang (1998) and Chari and Kehoe (1990). We strengthen the equilibrium concept, following Gorton, He and Huang (2014), so that we can work with functions. This may be of independent interest.

The dynamics in this economy are quite different from the standard model (without macroprudential policy). Here, the central bank trades one form of money for another to balance between the safe public collateral, Treasuries, and the fragile private collateral, MBS, as well as worrying about inflation. As discussed above, if the central bank does not inject enough Treasuries into the economy, the private sector generates more MBS. A change in Treasuries is negatively correlated with the production of MBS. At the same time, the central bank's money supply affects the price level, which affects the real value of the liquidity. A high initial money supply drives up the price level today, which decreases the real value of the liquidity for production; to balance that, the central bank needs to create more liquidity in nominal terms by buying Treasuries through open market operations, which drives up the amount of MBS and the money supply. However, if the ratio of MBS to Treasuries is too high, this cannot be optimal due to the prohibitively high risk of financial fragility. Therefore, with a high initial money supply, it may be optimal for the central bank to reduce private liquidity in nominal terms by selling Treasuries through open market operations, which reduces the amount of MBS, the money supply and the price level. In this way, the real value of liquidity and output may drop, but not as much as the welfare loss of financial fragility avoided.

In Section 2 we start the analysis by specifying and solving a two period model. We then assume specific functional forms and examine monetary policy. Section 3 presents and solves the infinite horizon model. The conclusion is Section 4.

2. The Two-Period Model

We begin with a two-period model in order to convey the basic setting. In the two-period model there is no commitment problem for the central bank, as will be seen. We first explain the model and then discuss the assumptions.

2.1 Model Setup

There are two types of goods available each period, perishable cash goods, which are subject to a cash-in-advance constraint (CIAC), and credit goods, which are housing services, at prices p and q, respectively. There is a continuum (of measure 1) of infinitely–lived identical agents. They consume cash goods and houses, and they also own the proceeds from production and home rental. Agents are arranged on a circle. Each agent is renting a house from the agent on the left and renting out a house he owns to an agent on the right. Each agent will purchase housing services and receive payments from the rental of housing services. Think of goods as being of different colors. Agents prefer the color of the goods produced by the agents to their left. For markets to be competitive there must be a double

[4] On the computation issues see Judd, Yeltekin and Conklin (2003) and Sleet and Yeltekin (2002).

continuum. But, because agents are otherwise identical we will speak of a representative agent. The central bank conducts monetary policy through trading in open market operations, adjusting the quantity of cash (M) and Treasuries (TB) in the economy.

The supply of houses, H, is a constant, and each period one unit of house generates one unit of housing services. Units of houses can be rented out at a price, q per unit. Define Q to be the market value of one unit of a house. In the background, the agent will obtain a variable rate mortgage to buy h units of house; the agent makes rQ in mortgage payments for each unit of house he buys, with a total mortgage payment rQh in each period. In equilibrium, renting one unit of housing services at price q is equivalent to buying one unit of house with a mortgage payment rQ, that is $q = rQ$ (or, equivalently, the house price is the annuity value of the mortgage payments).

Mortgage-backed-securities (MBS) are generated from housing sales, and the amount of MBS is proportional to the total amount of credit generated to finance housing sales, that is, $MBS = QH$.

Each agent owns a firm which produces cash goods using real Treasuries and real MBS as inputs. (This is discussed below.) The liquidity of MBS is not as good as Treasuries, since MBS are privately-produced; with respect to liquidity services MBS are only worth δMBS, where $\delta < 1$. The value of δ can be thought of as a haircut. It is a measure of market liquidity; the higher the value of δ, the better the market liquidity of MBS (and the more that can be borrowed). We assume the production function takes the following form, $y = f((TB_2 + \delta MBS)/p, \varepsilon)$ with $\delta < 1$. We interpret $(TB + \delta MBS)/p$ as the amount of liquidity (in real terms) provided through using Treasuries and mortgage-backed securities as collateral. Therefore, we define $L \equiv (TB + \delta MBS)/p$.

We now describe how the MBS are created (via securitization) and used in four steps (we only show the net changes to the balance sheets). For clarity we omit cash balances and Treasury holdings.

Step 1: An agent/firm borrows Qh from a bank to buy h units of a house. At the same time the agent/firm receives QH from the agent buying his house. He deposits this in the bank. In equilibrium, market clearing will require that $h = H$.

Bank		**Special Purpose Vehicle**		**Agent/Firm**	
Asset	Liability	Asset	Liability	Asset	Liability
Qh	QH	0	0	QH	Qh

Step 2: The bank securitizes the loan Qh through a special purpose vehicle, and the agent/firm uses its deposits to buy the MBS issued by the special purpose vehicle.

Bank		Special Purpose Vehicle		Agent/Firm	
Asset	Liability	Asset	Liability	Asset	Liability
0	0	*Qh*	*MBS*	*MBS*	*Qh*

Step 3: The agent/firm uses *MBS* as collateral to borrow δMBS from the bank for production.

Bank		Special Purpose Vehicle		Agent/Firm	
Asset	Liability	Asset	Liability	Asset	Liability
δMBS	δMBS	*Qh*	*MBS*	$MBS+\delta MBS$	$Qh+\delta MBS$

Step 4: The agent/firm produces and pays off all its debt (including the mortgage payment and the principal of debt); the special purpose vehicle passes the debt payment *Qh* to the agent/firm, who is the *MBS* holder, and the *MBS* are cleared; the bank receives δMBS loan repayments and pays off its own debt.

Bank		Special Purpose Vehicle		Agent/Firm	
Asset	Liability	Asset	Liability	Asset	Liability
0	0	0	0	0	0

The time line is as follows:

Date 1

1. The agent starts with m_1 and $w_1 = m_1 + tb_1(1 + r_0)$, and needs to pay a lump sum tax TB_1r_0
2. Output y_1 is realized; the central bank conducts open market operations, determines the aggregate supply of M_2 and TB_2, price p_1, q_1 and r_1 are formed
3. The agent receives income $p_1y_1 + qH$, and consumes c_1 and h, and ends up holding m_2 and tb_2 as savings

Date 2

1. The agent starts with m_2 and tb_2, with $w_2 = m_2 + tb_2(1 + r_1)$, and needs to pay a lump sum tax TB_2r_1
2. Output y_2 is realized, and the agent receives income p_2y_2
3. The agent consumes c_2, and chooses final wealth w_3
4. The agent pays a lump sum tax TB_2, and final consumption occurs

Note:

(i) The representative agent has wealth w_t at the beginning of period t, and pays a lump sum tax $TB_t r_{t-1}$. Define net worth as $nw_t \equiv m_t + tb_t(1 + r_{t-1}) - TB_t r_{t-1}$.

(ii) At the aggregate level, the sum of cash and Treasuries, net of the tax payment, is a constant, and we define $NW \equiv M_1 + TB_1 = M_2 + TB_2$.

(iii) At the end of date 1, each unit of house has a resale value of $Q' = q/r_1$ Therefore, we have the value of the house at the beginning of date 1, $Q = (q + Q')/(1 + r_1) = q/r_1$. This assumption makes the model consistent with the infinite horizon model.

(iv) Open market operations occur at date 1 and affect output at date 2. The quantity of Treasuries is taxed away and is irrelevant for the final consumption, by assumption.

(v) Final consumption is the real value of final wealth minus the tax payment, normalized by the date 2 cash goods price. The composition of final wealth does not affect utility, by assumption. Specifying the final consumption in this way closes the model.

At date 1, given the cash goods price p_1, the housing services price q, and the interest rate r_1, the representative agent is choosing cash goods consumption, c_1, housing services consumption, h, a cash amount, m_2, and Treasuries, tb_2, subject to the budget constraint and the cash-in-advance constraint.

$$\begin{aligned} &\max_{c_1, h_1, m_2, tb_2} u(c_1, h) + \beta U(m_2, tb_2) \qquad (1) \\ &s.t. \quad p_1 c_1 + qh + m_2 + tb_2 \le p_1 y_1 + qH + nw_1 \\ &\qquad\;\; p_1 c_1 \le m_1 . \end{aligned}$$

In the above optimization problem, $U(m_2, tb_2)$ is the utility generated from date 2 consumption, as a function of the individual state variables, m_2 and tb_2.

At $t = 2$, the representative agent starts with wealth $w_2 = m_2 + tb_2(1 + r_1)$, pays a lump sum tax $TB_2 r_1$, and receives income from cash goods output y_2 and housing services supplied, H. With $MBS = QH$, we have $y_2 = f((TB_2 + \delta MBS)/p_1, \varepsilon)$. The agent also pays a lump sum tax TB_2 from his final wealth w_3. Given the cash goods price, p_2, the representative agent is choosing cash goods consumption, c_2, and final wealth, w_3, subject to the budget constraint and the cash-in-advance constraint.

$$\begin{aligned} &U(m_2, tb_2) = E\left[\max_{c_2, w_3} [v(c_2) + \alpha v(c_3)]\right] \\ &s.t. \quad p_2 c_2 + w_3 \le p_2 y_2 + nw_2 \\ &\qquad\;\; p_2 c_2 \le m_2 \\ &\qquad\;\; c_3 = nw_3 / p_2 = (w_3 - TB_2)/ p_2 \qquad (2) \\ &\qquad\;\; nw_2 = m_2 + tb_2(1 + r_1) - TB_2 r \\ &\qquad\;\; y_2 = f((TB_2 + MBS)/ p_1, \varepsilon) \end{aligned}$$

Note that housing consumption at date 2 plays no role (there is simply a residual value to the house, Q'), we have omitted it from (2).

We now turn to the central bank's objective function. Subject to a boundary condition on expected deflation (discussed below), the central bank is choosing M_2 to maximize social welfare, which contains two parts: (i) The expected utility of the representative agent from consumption; (ii) The welfare loss due to a possible financial crisis, which—as discussed in the Introduction (and further below) —is a function of the ratio of MBS/Treasuries at date 1, and which is also a function of the aggregate state variable, M_2. We refer to the likelihood of a financial crisis as "financial fragility" henceforth.

The central bank's choice of the aggregate variable, M_2, affects social welfare through several channels: (i) Supply channel: M_2 affects the level of p_1 (if the cash-in-advance constraint is not binding at date 1), TB_2 and MBS (through q), and consequently affects output at date 2 (because p_1 affects real liquidity); (ii) Demand channel (price effect): M_2 affects the price level p_1 (if the cash-in-advance constraint is not binding at date 1), q and p_2 (if the cash-in-advance constraint is binding at date 2), as well as the interest rate level r_1, and therefore affects the consumption and saving behavior of the agent; (iii) Demand channel (wealth effect): output is also one source of income for an agent; (iv) Externality: the ratio of MBS/Treasuries at date 1 determines the welfare loss due to its effect on financial fragility.

The optimization problem of the central bank can be written is:

$$\begin{aligned} &Max_{M_2} \{u(c_1, h_1) + \beta U(M_2, TB_2) - \psi(MBS / TB_2)\} \\ &s.t. \quad E[p_2 / p_1] \geq \underline{\pi} \end{aligned} \tag{3}$$

The constraint on inflation means that the central bank cannot have $1 in the economy with an implicitly very high velocity. The constraint basically says that velocity is fixed.

The central bank's objective function is concerned with the effects of money (monetary policy), but it also conducts macroprudential policy via the ψ function. It is concerned about managing financial fragility (the likelihood of a financial crisis). For a given amount of Treasuries, production of MBS constitutes a credit boom and has an externality that private agents do not take into account; it raises the likelihood of a crisis. The welfare loss ψ is a function of the ratio of MBS to TB_2. When relatively more mortgage-backed securities are used as collateral for production (a credit boom), the probability of financial fragility increases, and we use the reduced form function ψ to capture the expected welfare loss from the likelihood of financial fragility. In equilibrium there will never be a financial crisis, but the objective function of the central bank specifies that welfare changes are smoothly changing with the ψ function. In other words, it is painful for the central bank to have a higher likelihood of crisis, since it does not know exactly what ratio of MBS to Treasuries will result in a crisis.

The representative agent, being small, does not take the possibility of financial fragility into account. Only the central bank can internalize the welfare loss due to the likelihood of financial fragility by choosing the aggregate level of the money supply. The representative agent cannot affect the aggregate money supply, and he behaves as if the welfare loss due to financial fragility does not exist, though he suffers if there is a financial crisis.

We will focus on symmetric equilibrium in which all agents start with the same endowment and behave the same way. In general, there can be many equilibria for a strategic game between the central bank and the continuum of small agents. We focus on the sequential equilibrium in which after observing every value of M_2, all agents will behave competitively and rationally. We define the sequential equilibrium below.

Definition 1 (Sequential Equilibrium): A sequential equilibrium satisfies the following two conditions:

(i) Given the central bank's choice of M_2, and the realization of cash goods output at date 2, $\{p_1, q, r_1, p_2\}$ and $\{c_1, h, m_2, tb_2, c_2, w_3\}$ are the competitive equilibrium of this economy, that is: Given $\{p_1, q, r_1, p_2\}$, $\{c_1, h, m_2, tb_2, c_2, w_3\}$ solves (1) and (2), and $\{p_1, q, r_1, p_2\}$ are such that markets are cleared, with $c_1 = y_1$, $h = H$, $m_2 = M_2$, $tb_2 = TB_2 = NW - M_2$, $c_2 = y_2$, and $w_3 = M_2 + TB_2$.

(ii) The central bank's choice of M_2 maximizes social welfare.

2.2 Discussion of the Model

Housing plays a central role in the model because the privately-produced collateral is mortgage-backed securities the amount of which is endogenously determined. This is consistent with the importance of housing for the macroeconomy, see, e.g., Jorda, Schularick and Taylor (2014) and Leamer (2007), and it is consistent with housing being at the center of financial crises.

A key ingredient of the model is the demand for Treasuries and MBS. Putting Treasuries and MBS in the production function is a reduced form for the use of Treasuries and MBS as collateral in the economy. It is simpler than using a "collateral in advance" constraint.[5] Privately-produced collateral, MBS, are not as liquid as Treasuries, hence the $\delta < 1$ parameter.

As discussed in the Introduction, financial crises tend to occur when there is insufficient government debt in the economy, resulting in the private sector creating a substitute, but inferior form of collateral, MBS—a credit boom. The credit boom is an externality because it creates financial fragility. Private agents cannot affect the quantities of Treasuries and MBS in the economy. This becomes the job of the central bank, as represented here by the ψ function.

Note the timing of the open market operations. Agents enter the first period with m_1 and tb_1, but before transacting in the first period, the central bank conducts open market operations.

[5] There is an older literature on money in the production function. The issue first arose in monetary growth models, e.g., Levhari and Patinkin (1968) and the debate evolved from that point. Examples include Friedman (1969), Fischer (1974), and Saving (1972). Examples of the empirical literature include Sinai and Stokes (1972) and You (1981). Nguyen (1986) reviews a lot of the literature. Benchimol (2010) is a recent example.

This affects first period decisions via prices. Agents then enter period 2 with m_2 and tb_2. So, in the two-period model, there is no commitment problem, as there will be in the infinite horizon model. Still, in the two-period model, the central bank acts strategically, as a large player.

Note that the MBS implicitly pay interest. In the background, every agent is borrowing money to buy a house, paying a variable interest rate, while depositing all the money from selling his own house and receiving a variable interest rate via the MBS. So interest payments on the mortgages and the MBS cancel out.

We assume a constant housing supply for simplicity, but note that the price of houses can change. Also, housing services are proportional to the housing stock (with proportionality one) for simplicity. Note that in order to pay off his mortgage, an agent sells the house. But, the new housing price may be such that he defaults on his mortgage. For simplicity this is costless.

The lower bound on deflation corresponds to velocity being fixed, so that the central bank must supply a sufficient amount of money so that it can be used as a medium of exchange.

2.3. Equilibrium Characterization

2.3.1 Individual Agent Optimization in a Competitive Equilibrium

To simplify the analysis below (for the two period model), we assume there is no uncertainty in production, that is, $y_2 = f((TB_2 + \delta MBS)/p_1)$, and we also assume:

$$u(c_1, h_1) = \ln c_1 + \ln h_1$$
$$v(c_2) = \ln c_2, v(c_3) = \ln c_3.$$

We will start by solving the model backwards starting from t =2. In the second period there are no central bank actions, so it is a competitive equilibrium.

Lemma 1: In a sequential equilibrium, at date 2 the cash-in-advance constraint (CIAC) is always binding for the representative agent, i.e. $p_2 = M_2 / y_2$.

Proof: See Appendix 1. ■

This determines the price in the second period. With the price at date 2, we can solve the decision problem of an agent holding m_2 and tb_2 (recall that at date 2 housing has been omitted for simplicity).

Lemma 2: For an agent with holdings m_2 and tb_2, whether on-the-equilibrium path or off-the-equilibrium path at date 2, when

$$m_2 < nw_3 / \alpha \text{ with } nw_3 = M_2 + (tb_2 - TB_2)(1 + r_1)$$

then the CIAC is not binding and the individual agent's expected utility from consumption is:

$$U(m_2,tb_2)=(1+\alpha)\ln(f)+\ln(m_2/M_2)+\alpha\ln(nw_3/M_2);$$

And when

$$m_2 > nw_3/\alpha \text{ with } nw_3=\frac{\alpha}{1+\alpha}\left[M_2+m_2+(tb_2-TB_2)(1+r_1)\right]$$

then the CIAC is binding and the individual agent's expected utility is:

$$U(m_2,tb_2)=(1+\alpha)\ln(f)+(2+\alpha)\ln(nw_3/M_2)-\ln(\alpha).$$

Proof: See Appendix 1. ■

Remark: On-the-equilibrium path (the agent holds M_2 and TB_2), CIAC is always binding at date 2. We calculate the off-equilibrium path behavior, i.e., an agent holds any m_2 and tb_2, to calculate $U(m_2, tb_2)$ and $U_m(m_2, tb_2)$ and $U_{tb}(m_2, tb_2)$, which are used in Lemma 3 below.

It is easy to check that, at date 2, if the agent starts with $m_2 = M_2$ and $tb_2 = TB_2$, then the cash-in-advance constraint is binding, and we have $nw_3 = M_2$. The expected utility for the agent is $U(M_2,TB_2)=(1+\alpha)\ln(f)$, which is the expected utility for the representative agent.

Now we turn to the date 1 problem. We have the following lemma.

Lemma 3: In a sequential equilibrium, at date 1, if the central bank chooses M_2 such that $M_2 < M_1\beta$, then the cash-in-advance constraint is *not binding* for the representative agent, and we have:

$$p_1=\frac{M_2}{\beta y_1},\ \ q=\frac{M_2}{\beta H},\ \ 1+r_1=\frac{1}{\alpha}.$$

If the central bank chooses M_2 such that $M_2 > M_1\beta$, then the cash-in-advance constraint *is binding* for the representative agent, and we have:

$$p_1=\frac{M_1}{y_1},\ \ q=\frac{M_2}{\beta H},\ \ 1+r_1=\frac{1}{\alpha}.$$

Proof: See Appendix 1. ■

In the proof of Lemma 3, one of the key steps was to use $U(m_2, tb_2)$ from Lemma 2, and in particular, the derivatives with respect to m_2 and tb_2 were used to calculate the marginal value of money and the marginal value of Treasuries for the representative agent. $U(m_2, tb_2)$ should be interpreted as the expected utility of an agent with m_2 and tb_2 while everyone else is holding M_2 and TB_2, and this is the key feature of a small agent. Notice that, $U_m(m_2=M_2, tb_2=TB_2)$ and $U_{tb}(m_2=M_2,\ tb_2=TB_2)$ are in general not the same as $U_M(m_2 = M_2,\ tb_2 = TB_2)$ and $U_{TB}(m_2 = M_2,\ tb_2 = TB_2)$. To see this note that using the functional form of $U(m_2, tb_2)$ on-the-equilibrium path from Lemma 2 we have:

$$U(m_2, tb_2) = (1+\alpha)\ln(f) + \ln(m_2 / M_2) + \alpha \ln(nw_3 / p_2)$$
$$\text{with } nw_3 = M_2 + (tb_2 - TB_2)(1 + r_1).$$

and

$$U_m(m_2 = M_2, tb_2 = TB_2) = 1/M_2$$
$$U_{tb}(m_2 = M_2, tb_2 = TB_2) = \alpha(1+r_1)/M_2$$
$$U_M(m_2 = M_2, tb_2 = TB_2) = \frac{1+\alpha}{f} f_M$$
$$U_{TB}(m_2 = M_2, tb_2 = TB_2) = \frac{1+\alpha}{f} f_{TB} = -\frac{1+\alpha}{f} f_M.$$

We can see that they are very different. Intuitively, an individual agent is a price-taker, and he can only change his own choices of m_2 and tb_2 without affecting the aggregate level of M_2 and TB_2, while the central bank can change the aggregate economic variables. In the infinite horizon model, in general, the functional form $U(m_2, tb_2)$ usually cannot be derived analytically, however, we can show that the value of its derivatives with respect to m_2 and tb_2 at $m_2 = M_2$ and $tb_2 = TB_2$ are sufficient to solve for the symmetric equilibrium in which all agents hold the same amounts of m_2 and tb_2. So, instead of knowing the functional form of $U(m_2, tb_2)$, we only need to know two values, which reduces the complexity of the problem substantially as will be seen.

Following Lemma 2, we can again write the date 1 expected utility of an agent with m_1 and tb_1 as a function of m_1 and tb_1 which we omit to save space. We can also check that given the prices and the interest rate derived in Lemma 3, when an agent holds $m_1 = M_1$ and $tb_1 = TB_1$, he will optimally choose $m_2 = M_2$ and $tb_2 = TB_2$, and consumes $c_1 = y_1$ and $h = H$.

2.3.2 Optimal Monetary Policy

The central bank trades through open market operations to maximize social welfare:

$$V = u(y_1, H) + \beta U(M_2, TB_2) - \psi(MBS/TB_2)$$
$$= \ln(y_1) + \ln(H) + \beta(1+\alpha)\ln(f) - \psi(MBS/TB_2).$$

With $MBS = QH = qH / r_1 = \alpha M_2 / \beta(1-\alpha)$, define $L = (TB_2 + \delta MBS)/ p_1$ and C = $\ln(y_1)+\ln(H)$, and the optimal monetary policy solves the following optimization problem:

$$\max_{M_2} V(M_2) \equiv \beta(1+\alpha)\ln(f(L)) - \psi(MBS/TB_2) + C$$

$$= \begin{cases} \beta(1+\alpha)\ln f\left(\dfrac{[\beta NW + (\delta\alpha/(1-\alpha)-\beta)M_2]y_1}{M_2}\right) - \psi\left(\dfrac{\alpha M_2}{\beta(1-\alpha)(NW-M_2)}\right) + C \text{ if } M_2 < M_1\beta \\ \beta(1+\alpha)\ln f\left(\dfrac{[\beta NW + (\delta\alpha/(1-\alpha)-\beta)M_2]y_1}{\beta M_1}\right) - \psi\left(\dfrac{\alpha M_2}{\beta(1-\alpha)(NW-M_2)}\right) + C \text{ if } M_2 > M_1\beta. \end{cases} \quad (4)$$

We can see that the decision problems of the central bank are quite different depending on whether the cash-in-advance constraint is binding or not. When the central bank chooses a low money supply, M_2, the value of money is high for date 2 (because the price is low), agents are hoarding cash in hand and the price level at date 1 is low, and the cash-in-advance constraint is *not* binding. In this case we will say that the economy is in *recession* (the CIAC is not binding). When the central bank chooses a high money supply, M_2, the value of money is low at date 2, agents spend all their cash on hand and the price level is high at date 1, the cash-in-advance constraint is binding. In this case we say that the economy is *booming* (the CIAC is binding)

Assuming the production function has the simple linear form with $f(L) = AL$, we have the following:

Proposition 1: With $\delta\alpha/(1-\alpha) > \beta$, the amounts of real collateral and output at date 2 are decreasing with M_2 during recession and increasing with M_2 when the economy is booming.

Proof: See Appendix 1. ■

Proposition 1 shows the key link between the money supply and output; it works through the real value of collateral. And, this depends on whether the cash-in-advance constraint is binding or not.

In terms of the inflation, we have the following lemma:

Lemma 4 (Inflation is a Monetary Phenomenon): The rate of inflation between date 1 and date 2 is always increasing with the money supply M_2 chosen by the central bank regardless of whether the economy is booming (CIAC binding) or in recession (CIAC not binding) at date 1.

Proof: See Appendix 1. ■

Intuitively, when the economy is in recession, the increase in the money supply causes inflation, i.e. causes both the date 1 price, p_1, and the date 2 price, p_2, to increase, and the rise in p_1 results in lower real liquidity and hence lower output at date 2, and this further drives up the date 2 price, p_2.

When the economy is booming, the cash-in-advance constraint is binding and the price at date 1 is not affected by the money supply chosen by the central bank, while the price at date 2 increases as the money supply increases, i.e., driving inflation up.

For future use, we define M_π as the money supply such that $\pi(M_\pi) \equiv \underline{\pi}$, and we have:

$$M_\pi \equiv \frac{A\underline{\pi}\beta NW}{\beta - A\underline{\pi}(\delta\alpha/(1-\alpha)-\beta)}.$$

Notice that, both the inflation rate π and M_π are independent of date 1 output, y_1, and the initial money supply, M_1 (this is driven by the model assumptions). Recall that $NW = TB_2 + M_2$ is constant.

The welfare loss from financial fragility, ψ, is increasing with the ratio of MBS to Treasuries, and, in particular, it takes the following form for tractability, with $\gamma > 0$ being some constant:

$$\psi\left(\frac{\alpha M_2}{\beta(1-\alpha)(NW - M_2)}\right) = \gamma[\ln NW - \ln(NW - M_2)]$$

Recall that $TB_2=NW-M_2$.

Lemma 5 (Money Supply and Minimizing Financial Fragility): Let

$$M_\psi \equiv \frac{\beta(1+\alpha)(\delta\alpha/(1-\alpha)-\beta)-\gamma\beta}{\beta(1+\alpha)(\delta\alpha/(1-\alpha)-\beta)+\gamma(\delta\alpha/(1-\alpha)-\beta)} NW.$$

Assume $M_\psi > M_\pi$. Then for any M_1 satisfying $M_\psi > M_1\beta$, $M_2 = M_\psi$ yields the highest welfare for any $M_2 > M_1\beta$.

Proof: See Appendix 1. ■

Notice that M_ψ is also a constant independent of date 1 output, y_1, and initial money supply, M_1.

With the results in Proposition 1 and Lemmas 4-5, we can fully characterize the optimal money supply, which is stated in the following Proposition.

Proposition 2 (Optimal Monetary Policy): (i) If M_1 satisfies $M_\psi > M_\pi > M_1\beta$, then the optimal money supply is M_ψ; (ii) If M_1 satisfies $M_\psi > M_1\beta > M_\pi$, then the optimal money supply is M_π or M_ψ, whichever gives the highest value of social welfare; (iii) If M_1 satisfies $M_1\beta > M_\psi > M_\pi$, the optimal money supply is M_π.

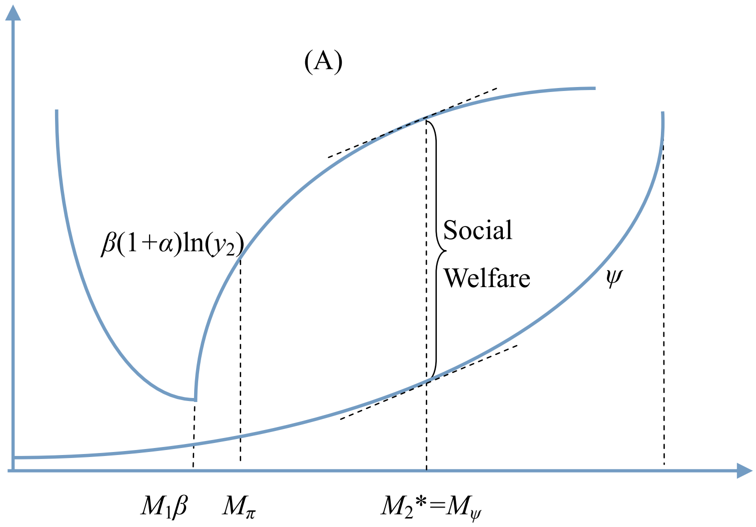
Proof: See Appendix 1. ■

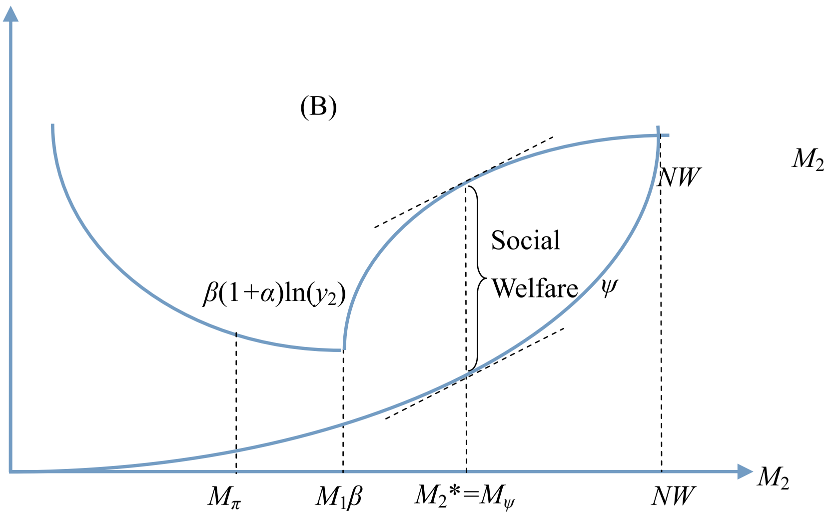
When the economy is in recession, output is decreasing with the money supply, and the welfare loss from financial fragility is increasing with the money supply. Therefore, the optimal money supply is at the lower bound.

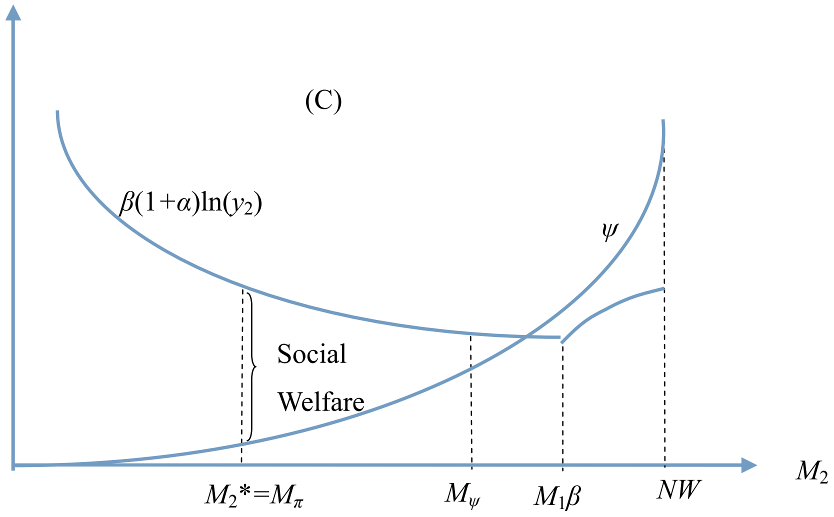
When the economy is booming, output is increasing with the money supply, but the welfare loss from financial fragility is also increasing with the money supply. Therefore, the optimal money supply balances the gain from the output increase with the welfare loss due to financial fragility.

Therefore, when the economy is in recession, it has a deflation and a low welfare loss due to financial fragility. When the economy is booming, it has high inflation and a high welfare loss due to financial fragility.

Examples of the optimal monetary policy are plotted in the figures below. In the figures the x-axis is M_2 and the y-axis is social welfare. Figure (A) for the case $M_\psi > M_\pi > M_1\beta$, Figure (B) for the case $M_\psi > M_1\beta > M_\pi$, Figure (C) for the case $M_1\beta > M_\psi > M_\pi$. In the figures the upper curve is the utility from output, and the bottom curve is the disutility from financial fragility. The dashed lines are the marginal utility.







Corollary 1: If M_ψ is the optimal money supply, then the optimal money supply decreases as the risk of financial fragility, as measured by γ, rises.

Proof: Algebra, thus omitted. ■

Intuitively, with a higher marginal cost of financial fragility, the central bank wants to lower the money supply, and this reduces the real value of collateral and increases the marginal utility from cash-goods consumption.

Corollary 2: There exists a cutoff value of the initial money supply M_1*, above which the economy is in recession, and M_π is the optimal money supply, and below which the economy is booming, and M_ψ is the optimal money supply.

Proof: See Appendix 1. ■

The above corollary tells us that when the initial money supply is high, it is more likely the economy is going into a recession with deflation and low welfare loss due to financial fragility. This boom-bust pattern is the equilibrium outcome under the optimal monetary policy.

2.4 Summary

The two-period model shows optimal monetary policy in an economy where collateral is central and the central bank seeks to mitigate the likelihood of a financial crisis. Monetary policy works through the real value of collateral. The two-period model, however, is simplified because the cash-in-advance constraint is always binding in the second period and there is no commitment issue for the central bank. In the second period there is no further action by the central bank and there are expectations about the future path of monetary policy.

3. Infinite Horizon Model

We now turn to the infinite horizon model. In this setting, there is a commitment problem; agents do not know the future path of monetary policy. Also, the cash-advance-constraint in the future may or may not be binding.

3.1. Model Set-Up

At the beginning of each period $t \geq 1$, the representative agent starts with w_t units of wealth, which includes m_t units of cash and $tb_t(1+r_{t-1})$ units of Treasuries, but the agent needs to pay a lump sum tax, $TB_t r_{t-1}$, to the government/central bank. (For simplicity, the principal is not repaid.)

At time t, the representative agent is consuming two types of goods, perishable cash goods, which are subject to a cash-in-advance constraint, and nonperishable credit goods, housing services rental, at prices p_t and q_t, respectively. Define Q_t to be the market value of the house at time t. Equivalent to rental cash flows, the agent borrows a variable rate loan to buy a house, and the agent makes $q_t = r_t Q_t$ of mortgage payment for each unit of housing purchased in each period. The MBS_t are generated from housing sales, that is, the bank lending, and the amount of MBS_t, generated through securitization, is equal to the total amount of credit generated, that is, $MBS_t = Q_t H$. The representative agent owns the proceeds from cash goods output, $p_t y_t$, as well as the income from renting out his house, $q_t H$.

Each agent owns a firm that uses real Treasuries and real *MBS* as collateral for production. The liquidity of *MBS* is not as good as Treasuries; with respect to liquidity services they are only worth δMBS, where $\delta < 1$. The value of δ can be thought of as a haircut. It is a measure of market liquidity; the higher the value of δ, the better the market liquidity of *MBS*. Therefore, cash goods output, y_t, is a function of the real amount of Treasuries (TB_t) chosen last period and mortgage-backed-securities (MBS_{t-1}) from last period and the current shock, ε_t, i.e. $y_t = f((TB_t + \delta MBS_{t-1})/p_{t-1}, \varepsilon_t)$ with ε_t being i.i.d.

At time t, the agent trades in the open market operations with the central bank, and after trading he ends up with m_{t+1} units of cash and tb_{t+1} units of Treasuries. The central bank is conducting monetary policy through open market operations, adjusting the quantity of cash (M_{t+1}) and Treasuries (TB_{t+1}) in the economy.

The time line for period t is as follows:

1. At the beginning of period t (date t), the agent starts with m_t and $tb_t(1+r_{t-1})$, but needs to pay a lump sum tax $TB_t r_{t-1}$;
2. Output y_t is realized, and The central bank conducts open market operations, which determines the aggregate supplies of M_{t+1} and TB_{t+1}; prices p_t, q_t and r_t are formed;
3. The agent receives income $p_t y_t + q_t H$, and chooses the consumption of cash goods, c_t, and housing services, h_t, subject to the budget constraint and the cash-in-advance

constraint, generating utility $u(c_t, h_t)$, while holding m_{t+1} and tb_{t+1} as savings. Remarks: The lump sum tax is used for interest payments on outstanding Treasuries. We assume the central bank/government will tax the exact amount to cover the interest payments. After paying the lump sum tax, the representative agent carries a net wealth of $nw_t \equiv m_t + tb_t(1+r_{t-1}) - TB_t r_{t-1}$ into the beginning of period t, and he also receives income $p_t y_t + q_t H$. On-the-equilibrium path $nw_t = M_t + TB_t = NW$, which is a constant as we assume the only monetary policy allowed in the model are the open market operations, which involve a one-to-one exchange of cash and Treasuries.

The representative agent's lifetime utility from consumption can be written as $E\left[\sum_{t=1}^{\infty} \beta^t u(c_t, h_t)\right]$, where $u(.)$ is the utility function, and $\beta \in (0, 1)$ is the discount factor. We assume that $u(.)$ is increasing and concave in both c and h, that is, $u_c > 0$, $u_{cc} \leq 0$, $u_h > 0$, and $u_{hh} \leq 0$. The central bank is maximizing social welfare, which is the utility from the agent's consumption net of the welfare loss from financial fragility, which is a function of the ratio of mortgage-backed securities to Treasuries, $E\left[\sum_{t=1}^{\infty} \beta^t u(c_t, h_t) - \psi(MBS_t / TB_{t+1})\right]$, where $\psi(.)$ is a convex function increasing in MBS_t / TB_{t+1}. The term $\psi(MBS_t / TB_{t+1})$ affects the welfare of each agent in equilibrium, but it does not affect their optimization problem as it is a function of aggregate variables only.

In period t, after output y_t is realized, the central bank chooses the amount of Treasuries to trade, ΔTB_{t+1}, which leads to $TB_{t+1} = TB_t + \Delta TB_{t+1}$ and $M_{t+1} = M_t - \Delta TB_{t+1}$, and the representative agent chooses consumption of cash goods, c_t, consumption of housing services, h_t, cash holdings m_{t+1}, and Treasury bill holdings, tb_{t+1}, as functions of the prices in the economy, p_t, q_t and r_t. The equilibrium prices p_t, q_t and r_t are such that all markets clear, that is, $c_t = y_t$, $h_t = H$, $m_{t+1} = M_{t+1}$, and $tb_{t+1} = TB_{t+1}$.

Similar to the two-period case, we will study sequential equilibria with public strategies, in which the strategies of both the central bank and the representative agent depend only on public information. This type of equilibrium is called Perfect Public Equilibrium (PPE) (see Fudenberg, Levine, and Maskin (1994)). The public history at time t is denoted as η^t:

$$\eta^t = \begin{cases} \{M_1, TB_1, y_1, r_0\} & for\ t = 1 \\ \{M_1, TB_1, y_1, r_0\} \cup \{M_{\tau+1}, TB_{\tau+1}, y_{\tau+1}, p_\tau, q_\tau, r_\tau\}_{\tau=1}^{t-1} & for\ t > 1 \end{cases}.$$

The representative agent's strategy can be written as:

$$\sigma_a = \{c_t(\eta^t, p_t, q_t, r_t), h_t(\eta^t, p_t, q_t, r_t), m_t(\eta^t, p_t, q_t, r_t), tb_t(\eta^t, p_t, q_t, r_t)\}_{t=1}^{\infty}.$$

The central bank's strategy can be written as:

$$\sigma_b = \{TB_t(\eta^t)\}_{t=1}^{\infty} .$$

We can see that the only state variable of the economy is the money supply in the economy, and we denote the economy that started with M_1 as $\Phi(M_1)$. A strategy profile for the economy $\Phi(M_1)$ is denoted as $\sigma = (\sigma_a, \sigma_b)$. We will use the Strong Markov Perfect Public Equilibrium concept to define our equilibrium, as in Gorton, He and Huang (2014), which is defined below. We will first construct an auxiliary competitive equilibrium by assuming that the central bank adopts an exogenous strategy σ_b. This auxiliary equilibrium will be useful subsequently because along the equilibrium path private agents behave as if the central bank has an exogenous strategy. So, subsequently determined equilibria must be in the set of equilibria for this auxiliary problem.

3.2. An Auxiliary Competitive Equilibrium

Given the central bank's exogenous strategy σ_b, the representative agent is solving the following optimization problem:

$$\begin{aligned} &\max E[\textstyle\sum_{t=1}^{\infty} \beta^t u(c_t, h_t)] \\ &s.t. \quad \begin{aligned} &p_t c_t + q_t h_t + m_{t+1} + tb_{t+1} \le p_t y_t + q_t H + w_t - TB_t r_{t-1} \\ &p_t c_t \le m_t \end{aligned} \end{aligned} \qquad (3)$$

Given the realization of cash goods output, the competitive equilibrium of this economy (conditional on the central bank's strategy), denoted as $\Phi(M_1|\sigma_b)$, is a sequence of cash goods prices, housing services prices and interest rates, $\{p_t, q_t, r_t\}_{t=1}^{\infty}$ and a sequence of cash goods consumption, housing consumption, cash holding and Treasuries holding $\{c_t, h_t, m_{t+1}, tb_{t+1}\}_{t=1}^{\infty}$, such that:

1. Given $\{p_t, q_t, r_t\}_{t=1}^{\infty}$, $\{c_t, h_t, m_{t+1}, tb_{t+1}\}_{t=1}^{\infty}$ maximizes $E\left[\sum_{t=1}^{\infty} \beta^t u(c_t, h_t)\right]$ subject to the budget constraints and cash-in-advance constraints.

2. Markets are cleared, that is, $c_t = y_t$, $h_t = H$, $m_{t+1} = M_{t+1}$, and $tb_{t+1} = TB_{t+1}$, for any t.

In equilibrium, the representative agent consumes today and saves in the form of cash and Treasuries. To construct the auxiliary equilibrium, we need to define the marginal value of additional cash holding, X, and the marginal value of additional Treasuries holding, Z. Under the usual regularity conditions, the budget constraint is binding, but not necessarily the cash-in-advance constraint. We will define *X* and *Z* separately in the case when the cash-in-advance constraint is binding and in the case when that is not.

Case 1: If on-the-equilibrium path, the cash-in-advance constraint is binding, the additional cash holding today will increase cash goods consumption tomorrow, while the additional Treasuries holding today will increase housing consumption tomorrow, given everything else the same. We can write a one-period optimization problem as follows

$$
\begin{aligned}
&\max_{c_{t+1},h_{t+1}} u(c_{t+1},h_{t+1}) \\
&\quad p_{t+1}c_{t+1} + q_{t+1}h_{t+1} + m_{t+2} + tb_{t+2} = p_{t+1}y_{t+1} + q_{t+1}H + w_{t+1} - TB_{t+1}r_t \\
&s.t. \quad w_{t+1} = m_{t+1} + tb_{t+1}(1+r_t) \\
&\quad p_{t+1}c_{t+1} = m_{t+1}
\end{aligned}
$$

The binding cash-in-advance constraint and budget constraint give:

$$
\begin{aligned}
c_{t+1}^B &= m_{t+1} / p_{t+1} \\
h_{t+1}^B &= \left(p_{t+1}y_{t+1} + q_{t+1}H + w_{t+1} - TB_{t+1}r_t - p_{t+1}c_{t+1} - m_{t+2} - tb_{t+2}\right)/ q_{t+1} \\
&= \left(p_{t+1}y_{t+1} + q_{t+1}H + tb_{t+1}(1+r_t) - TB_{t+1}r_t - m_{t+2} - tb_{t+2}\right)/ q_{t+1}
\end{aligned}
$$

Define the marginal value of cash holding at time *t*+1 as follows

$$X_{t+1}^B = u_c(y_{t+1}, H) / p_{t+1},$$

which is just the first order condition of expected *t*+1 utility with respect to m_{t+1} at $c_{t+1}^B(w_{t+1}) = y_{t+1}$ and $h_{t+1}^B(w_{t+1}) = H$.

Similarly, define the expected marginal value of Treasury holdings at time *t*+1 as follows:

$$Z_{t+1}^B = u_h(y_{t+1}, H)(1+r_t) / q_{t+1},$$

which is just the first order condition of expected *t*+1 utility with respect to tb_{t+1} at $c_{t+1}^B(w_{t+1}) = y_{t+1}$ and $h_{t+1}^B(w_{t+1}) = H$.

For the cash-in-advance constraint to be binding, we need:

$$
\begin{aligned}
&u_c(y_{t+1}, H) / p_{t+1} > u_h(y_{t+1}, H) / q_{t+1} \\
&\Rightarrow X_{t+1}^B > Z_{t+1}^B / (1+r_t)
\end{aligned}
$$

Case 2: If on-the-equilibrium path, the cash-in-advance constraint is *not* binding, that is $p_{t+1}y_{t+1} < m_t$, then to calculate the additional utility from extra cash holdings and Treasury holdings, we write out a one-period optimization problem as follows:

$$\begin{aligned} &\max_{c_{t+1},h_{t+1}} u(c_{t+1},h_{t+1}) \\ s.t.\quad & p_{t+1}c_{t+1} + q_{t+1}h_{t+1} + m_{t+2} + tb_{t+2} = p_{t+1}y_{t+1} + q_{t+1}H + w_{t+1} - TB_{t+1}r_t \\ & w_{t+1} = m_{t+1} + tb_{t+1}(1+r_t) \end{aligned}$$

in which c_{t+1} and h_{t+1} are the only choice variables, and m_{t+2} and tb_{t+2} can be deemed as fixed. Denote the solution to the above problem as $c_{t+1}^{NB}(w_{t+1})$ and $h_{t+1}^{NB}(w_{t+1})$, which are functions of the initial wealth, w_{t+1}.

Now we can define the marginal value of cash holding at time t+1 as follows

$$\begin{aligned} X_{t+1}^{NB} &= u_c(c_{t+1}^{NB},h_{t+1}^{NB})\frac{dc_{t+1}^{NB}}{dw_{t+1}}\frac{\partial w_{t+1}}{\partial m_{t+1}} + u_h(c_{t+1}^{NB},h_{t+1}^{NB})\frac{dh_{t+1}^{NB}}{dw_{t+1}}\frac{\partial w_{t+1}}{\partial m_{t+1}}, \\ &= u_c(y_{t+1},H)c_{w,t+1}^{NB} + u_h(y_{t+1},H)h_{w,t+1}^{NB} \end{aligned}$$

which is again the first order condition of expected t+1 utility with respect to m_t at $c_{t+1}^{NB}(w_{t+1}) = y_{t+1}$ and $h_{t+1}^{NB}(w_{t+1}) = H$.

Similarly, we can define the expected marginal value of Treasuries holding at time t+1 as follows

$$\begin{aligned} Z_{t+1}^{NB} &= u_c(c_{t+1}^{NB},h_{t+1}^{NB})\frac{dc_{t+1}^{NB}}{dw_{t+1}}\frac{\partial w_{t+1}}{\partial tb_{t+1}} + u_h(c_{t+1}^{NB},h_{t+1}^{NB})\frac{dh_{t+1}^{NB}}{dw_{t+1}}\frac{\partial w_{t+1}}{\partial tb_{t+1}} \\ &= u_c(y_{t+1},H)c_{w,t+1}^{NB}(1+r_t) + u_h(y_{t+1},H)h_{w,t+1}^{NB}(1+r_t) \\ &= X_{t+1}^{NB}(1+r_t) \end{aligned}$$

which is again the first order condition of expected t+1 utility with respect to tb_{t+1} at $c_{t+1}^{NB}(w_{t+1}) = y_{t+1}$ and $h_{t+1}^{NB}(w_{t+1}) = H$.

Furthermore, given the binding budget constraint, it is easy to check that

$$\begin{aligned} & u_c(y_{t+1},H)/p_{t+1} = u_h(y_{t+1},H)/q_{t+1} \\ & p_{t+1}c_{w,t+1}^{NB} + q_{t+1}h_{w,t+1}^{NB} = 1 \end{aligned}$$

which implies

$$u_c(y_{t+1},H)c^{NB}_{w,t+1}+u_h(y_{t+1},H)h^{NB}_{w,t+1}=u_c(y_{t+1},H)/p_{t+1}=u_h(y_{t+1},H)/q_{t+1}$$

Therefore, we can further simplify the expressions of X^{NB}_{t+1} and Z^{NB}_{t+1} as follows:

$$\begin{aligned}X^{NB}_{t+1}&=u_c(y_{t+1},H)/p_{t+1}\\Z^{NB}_{t+1}&=u_c(y_{t+1},H)(1+r_t)/p_{t+1}=u_h(y_{t+1},H)(1+r_t)/q_{t+1}.\end{aligned}$$

Combining the two cases, we have:

$$\begin{aligned}X_{t+1}&=E_B[X^B_{t+1}]\Pr(B)+E_{NB}[X^{NB}_{t+1}]\Pr(NB)\\Z_{t+1}&=E_B[Z^B_{t+1}]\Pr(B)+E_{NB}[Z^{NB}_{t+1}]\Pr(NB)\end{aligned}$$

The variables X_{t+1} and Z_{t+1} represent the representative agent's marginal expected lifetime utility for an additional amount of cash and Treasuries, respectively, that he holds at the beginning of period t+1. These two variables summarize all the information the representative agent needs to make the consumption and saving decision at time t. In our two-period model, X_{t+1} and Z_{t+1} correspond to the derivative of $U(m_2, tb_2)$ with respect to m_2 and tb_2, respectively, at $m_2 = M_2$ and $tb_2 = TB_2$, which is the only information we need from date 2 when we characterize the date 1 problem for our two-period model in Section 2.2.2 (see the Lagrange condition).

Suppose we have solved the auxiliary competitive equilibrium for the economy $\Phi(M_1|\sigma_b)$, and we can then calculate the values of X_{t+1} and Z_{t+1} for every period. Let us construct a one-period economy where the representative agent owns initial cash holdings and Treasuries of m_t and tb_t and X_{t+1} and Z_{t+1} are taken as exogenously given. Since the central bank's trading strategy is now exogenously given, the representative agent in this one-period economy chooses cash goods consumption, c_t, housing services consumption, h_t, cash holdings, m_{t+1}, and Treasuries, tb_{t+1}, with an augmented utility function over consumption in cash goods and housing and end-of-period cash holding and Treasuries holding, $u(c_t,h_t)+\beta X_{t+1}m_{t+1}+\beta Z_{t+1}tb_{t+1}$, subject to the budget constraint and cash-in-advance constraint:

$$\begin{aligned}&\max\left[u(c_t,h_t)+\beta X_{t+1}m_{t+1}+\beta Z_{t+1}tb_{t+1}\right]\\ s.t.\quad &p_tc_t+q_th_t+m_{t+1}+tb_{t+1}\le p_ty_t+q_tH+w_t-TB_tr_{t-1}\\ &p_tc_t\le m_t\end{aligned}\qquad (4)$$

The competitive equilibrium of this static one-period economy consists of prices $\{p_t, q_t, r_t\}$ and consumptions plus savings $\{c_t, h_t, m_t, tb_t\}$ such that the following two conditions are satisfied:

1. Given $\{p_t, q_t, r_t\}$, $\{c_t, h_t, m_{t+1}, tb_{t+1}\}$ maximizes $u(c_t, h_t) + \beta X_{t+1} m_{t+1} + \beta Z_{t+1} tb_{t+1}$;

2. $\{p_t, q_t, r_t\}$ are such that $c_t = y_t$, $h_t = H$, $m_{t+1} = M_{t+1}$, and $tb_{t+1} = TB_{t+1}$.

Let CE(m_t, tb_t, X_{t+1}, Z_{t+1}) denote the set of competitive equilibrium allocations (c_t, h_t, m_{t+1}, tb_{t+1}) of this static one-period economy. We can show that the auxiliary competitive equilibrium of this economy is equivalent to a corresponding one-period economy constructed below with a transversality condition. As shown in Phelan and Stacchetti (2001), the transversality condition holds if we impose boundary conditions on the representative agent's marginal utility.

Proposition 3 (Equivalence of the Static One-Period Equilibrium to the Infinite Horizon Equilibrium): If the money supply is bounded from below away from zero, that is $0 < \underline{M} < M < NW$, output is bounded from above and below away from zero, that is $0 < \underline{y} < y < \overline{y} < \infty$, and $u(c,h)$ is concave with $0 < \underline{u}_c < u_c(c,h) < \overline{u}_c < \infty$ and $0 < \underline{u}_h < u_h(c,h) < \overline{u}_h < \infty$, then X_t and Z_t are bounded from above in a competitive equilibrium of the economy $\Phi(M_1 | \sigma_b)$. For $\{c_t^*, h_t^*, m_{t+1}^*, tb_{t+1}^*\}_{t=1}^{\infty}$ to be a competitive allocation of the economy $\Phi(M_1 | \sigma_b)$, a necessary and sufficient condition is that, for all t and η^t, $\{c_t^*, h_t^*, m_{t+1}^*, tb_{t+1}^*\}$ is the static one-period equilibrium outcome, that is, $\{c_t^*, h_t^*, m_{t+1}^*, tb_{t+1}^*\} \in CE(m_t, tb_t, X_{t+1}, Z_{t+1})$.

Proof: See Appendix 1. ■

With the above result, we can represent the infinite-horizon dynamic problem in recursive form, as shown below.

3.3. Strong Markov Perfect Public Equilibria

We now proceed to characterize the dynamic equilibrium with the central bank's decisions endogenized. For a given strategy profile $\sigma = (\sigma_a, \sigma_b)$, the representative agent has the following expected lifetime utility

$$U[\sigma] = E\left[\sum_{t=1}^{\infty} \beta^t u(c_t, h_t)\right].$$

The central bank has the following expected lifetime utility

$$V[\sigma]=E\left[\sum_{t=1}^{\infty}\beta^t\left(u(c_t,h_t)-\psi(MBS_t/TB_{t+1})\right)\right].$$

Definition 2 (Perfect Public Equilibrium) A strategy profile $\sigma = (\sigma_a, \sigma_b)$ is a Perfect Public Equilibrium (PPE) for the economy $\Phi(M_1)$ if for any $\tau \geq 1$, and history η^τ, the following conditions are satisfied:

1. Given the representative agent's strategy σ_a, the central bank has no incentive to deviate, that is, $V[(\sigma_{a\tau}, \sigma_{b\tau})] > V[(\sigma_{a\tau}, \sigma'_{b\tau})]$ for any $\sigma'_{b\tau} \neq \sigma_{b\tau}$, where $(\sigma_{a\tau}, \sigma_{b\tau})$ is the truncated equilibrium strategy profile $\sigma = (\sigma_a, \sigma_b)$ starting from $\tau \geq 1$;

2. $\{c_t, h_t, m_{t+1}, tb_{t+1}\}_{t=\tau}^{\infty}$ resulting from the representative agent's strategy $\sigma_{a\tau}$ is an auxiliary competitive equilibrium outcome of the economy $\Phi(M_\tau|\sigma_{b\tau})$.

The definition of PPE imposes two conditions. The first condition requires sequential optimality, that is, the central bank's continuation strategies must be best responses to the representative continuation strategies after any history η^τ. The second condition states the optimality of the representative agent's strategy in an auxiliary competitive equilibrium we analyzed in Section 3.1.

From the definition of PPE, we know that the continuation payoffs of a PPE after any history have to correspond to PPE profiles, so the lifetime expected payoffs can be factored into current payoffs and continuation PPE payoffs. As in Phelan and Stacchetti (2001), the recursive formalization involves not only the payoffs to the central bank and the representative agent, but also the marginal values of cash and Treasuries for the representative agent, which are the key features for the auxiliary competitive equilibrium. For any strategy profile $\sigma = (\sigma_a, \sigma_b)$, we define the marginal value of cash and Treasuries at the beginning of the game as

$$\begin{aligned} X[\sigma] &= E[u_c(c_1,h_1)/p_1] \\ Z[\sigma] &= E[u_h(c_1,h_1)(1+r_0)/q_1] \end{aligned}.$$

In Appendix 2 we show the recursive factorization of the defined PPE in terms of $V[\sigma]$, $X[\sigma]$ and $Z[\sigma]$, which can be replaced with simplified state-dependent value correspondences due to the existence of multiple PPEs. In our model, the state variable is the distribution of money holdings across agents, however, when all dispersed agents hold the same amount of money, the state variable degenerates to a single variable—the aggregate money supply in the economy, *M*. We define

$$(V(M), X(M), Z(M)) = \{(V[\sigma], X[\sigma], Z[\sigma]) \mid \sigma \text{ is a PPE for the economy } \Phi(M)\}.$$

The recursive formalization of the PPE only delivers value correspondences that depend on *M*, but the strategies of the central bank and the representative agent still depend on the history. For tractability, we restrict attention to strategies where the central bank's and the representative agent's strategies only depend on the state variable, *M*. These strategies are known as Markovian strategies. A Markov Perfect Public Equilibrium (MPPE) is a Perfect Public Equilibrium in which the central bank and the representative agent pay time-invariant Markovian strategies. As in Gorton, He and Huang (2013), we impose further restriction on the off-equilibrium-path strategies, and require off-equilibrium strategies to be the same as on-equilibrium strategies when the state variables are the same. Markovian strategies satisfying this consistency conditions are named as Strong Markov Perfect Public Equilibrium (SMPPE), which is formally defined below.

Definition 3 (Strong Markov Perfect Public Equilibrium) A Strong Markov Perfect Public Equilibrium (SMPPE) is a Markov Perfect Public Equilibrium (MPPE) that yields the same MPPE in every truncated continuation game regardless of on- or off-the-equilibrium path.

By imposing this restriction on off-equilibrium threats, we can study functions instead of correspondences. We can write an SMPPE as a set of functions, {*U*(*M*,*Y*), *V*(*M*, *y*), *X*(*M*, *y*), *Z*(*M*, *y*), *M'*(*M*, *y*), *p*(*M*, *M'*, *y*), *q*(*M*, *M'*, *y*), *r*(*M*, *M'*, *y*), *c*(*M*, *M'*, *y*), *h*(*M*, *M'*, *y*), *m'*(*M*, *M'*, *y*), *tb'*(*M*, *M'*, *y*)} that are derived from solving the optimization problems of the representative agent and the central bank.

$$\begin{aligned}
&U(M,y) = \max_{c,h,m',tb'}\{u(c,h) + \beta E_{y'}[X(M',y')]m' + \beta E_{y'}[Z(M',y')]tb'\} \\
&V(M,y) = \max_{M'}\{u(y,H) - \psi(MBS/TB') + \beta E_{y'}[V(M',y')]\} \\
&\qquad pc + qh + m' + tb' \le py + qH + nw \\
&\qquad pc \le m \\
&s.t. \quad nw = m + tb(1+r_0) - TBr_0 \\
&\qquad y' = f\left(\frac{TB' + \delta MBS}{p}, \varepsilon'\right)
\end{aligned} \tag{3}$$

where *nw* is equal to the initial wealth, *w*, net of the lump sum tax payment for the representative agent, *MBS* is equal to *QH* with *Q* = *q*/*r*, and r_0 is the interest rate from last period

Remarks: here we use { *U*(*M*,*Y*), *V*(*M*, *y*), *X*(*M*, *y*), *Z*(*M*, *y*)}, values after *y* is realized, instead of the expected values, { *U*(*M*,*Y*), *V*(*M*), *X*(*M*), *Z*(*M*)}. It turns out it is more convenient to state the results in terms of the realized values.

Substitute in the market clearing conditions with *m* = *M*:

$$
\begin{aligned}
&c(M, M'(M, y), y) = y \\
&h(M, M'(M, y), y) = H \\
&m'(M, M'(M, y), y) = M' = NW - TB'(M, y) \\
&tb'(M, M'(M, y), y) = TB'(M, y)
\end{aligned}
$$

and we have the following conditions that need to be satisfied:

1. *V*(*M*, *y*) is the value function of the central bank:

$$
\begin{aligned}
&V(M, y) = u(y, H) - \psi(MBS / TB') + \beta E_{y'}\left[V(M', y')\right] \\
&s.t. \quad y' = f\left(\frac{TB' + \delta MBS}{p}, \varepsilon'\right)
\end{aligned}
\tag{C1}
$$

2. *X*(*M*, *y*) is the marginal value of cash (*m*) for the representative agent:

$$
X(M, y) = \begin{cases} u_c(y, H)/p & \text{if} \quad u_c(y, H)/p > u_h(y, H)/q \\ u_c(y, H)c_w^{NB} + u_h(y, H)h_w^{NB} & \text{if} \quad py < M \end{cases}
\tag{C2}
$$

3. *Z*(*M*, *y*) is the marginal value of Treasuries (*tb*) for the representative agent:

$$
Z(M, y) = \begin{cases} u_h(y, H)(1 + r_0)/q & \text{if} \quad u_c(y, H)/p > u_h(y, H)/q \\ u_c(y, H)(1 + r_0)/p = u_h(y, H)(1 + r_0)/q & \text{if} \quad py < M \end{cases}
\tag{C3}
$$

4. Optimal cash holding, *m'*, for the representative agent:

$$
\begin{aligned}
&u_h(y, H)/q = \beta E_{y'}[X(M', y')] \\
&s.t. \quad y' = f\left(\frac{TB' + \delta MBS}{p}, \varepsilon'\right)
\end{aligned}
\tag{C4}
$$

5. Optimal Treasuries holding, *tb'*, for the representative agent:

$$
\begin{aligned}
&u_h(y, H)/q = \beta E_{y'}[Z(M', y')] \\
&s.t. \quad y' = f\left(\frac{TB' + \delta MBS}{p}, \varepsilon'\right)
\end{aligned}
\tag{C5}
$$

6. Optimal consumption of cash goods:

$$p=\begin{cases} M/y=(NW-TB)/y & \text{if } u_c(y,H)/p>u_h(y,H)/q \\ qu_c(y,H)/u_h(y,H) & \text{if } py<M \end{cases} \qquad \text{(C6)}$$

where *NW* is a constant that is equal to the sum of aggregate cash, *M*, and aggregate Treasuries, *TB*.

7. Optimal open market operations by the central bank:

$$\begin{aligned} &M'(M,y)=\arg\max_{M'}\{u(y,H)-\psi(MBS/TB')+\beta E_{y'}[V(M',y')]\} \\ &s.t. \quad y'=f\left(\frac{TB'+\delta MBS}{p},\varepsilon'\right) \end{aligned} \qquad \text{(C7)}$$

Remarks:

(i) With the binding budget constraint, we only have three first order conditions (C4-C6), and the optimal housing consumption is implied from the binding budget constraint. If the cash-in-advance constraint is also binding, then the first order condition in (C6) reduces to the binding cash-in-advance constraint.

(ii) The continuation values (or marginal values), *V*(*M′*, *y′*), *X*(*M′*, *y′*) and *Z*(*M′*, *y′*), in the above conditions reflect the consistency of the continuation game no matter whether it is on- or of-the-equilibrium path, and this is the key feature of SMPPE.

Proposition 4 (Existence of SMPPE) If *f*(*L*,*ε*) is continuous and *u*(*c*,*h*) is continuous and have continuous differentials, then there exists a Strong Markov Perfect Public Equilibrium.

Proof: See Appendix 1. ■

3.4. An Example with Policy Analysis

We assume *u*(*c*,*h*) = ln(*c*) + ln(*h*), and the production, *y′* = *f*(*L*,*ε*), has common support $Y=[\underline{y},\bar{y}]$. We also assume that the cost of financial fragility *ψ*(*ρ*) satisfies *ψ′*(*ρ*) > 0 (ρ≡MBS/TB) and the Inada conditions, that is, $\lim_{\rho\to 0}\psi'(\rho) = 0$ and $\lim_{\rho\to\infty}\psi'(\rho) = \infty$.

From (C6), we have:

$$py=\begin{cases} M=NW-TB & \text{if } \ M=py<qH \\ qH & \text{if } \ py<M \end{cases}$$

From (C2), we have:

$$X(M,y)=\begin{cases}1/py=1/M & \text{if} \quad M=py<qH\\ 1/py=1/qH & \text{if} \quad py<M\end{cases}$$

From (C3), we have:

$$Z(M,y)=\begin{cases}(1+r_0)/qH & \text{if} \quad M=py<qH\\ (1+r_0)/py=(1+r_0)/qH & \text{if} \quad py<M\end{cases}$$

From (C4), we have:

$$1/qH=\beta E[X(M',y')]$$

From (C5), we have:

$$1/qH=\beta E[Z(M',y')]$$

Again, we also impose a deflation bound and the central bank is solving the following problem:

$$\begin{aligned}&V(M,y)=\max_{M'} u(y,H)-\psi(MBS/TB')+\beta E[V(M',y')]\\ s.t.\quad &y'=f(L,\varepsilon)=f((TB'+\delta MBS)/p,\varepsilon)\\ &p'/p\geq\underline{\pi}\end{aligned}$$

Next, we are going to characterize some features of an SMPPE.

Lemma 6 (The Taylor Rule with Housing Rental Prices): Let g_q be the expected appreciation in home rental prices. The interest rate is equal to the expected home rental price appreciation rate plus one minus the time discounting rate i.e., $r\approx g_q+(1-\beta)$.

Proof: See Appendix 1. ■

This Taylor rule is derived from (C5) above with the assumed functional form, i.e., the FOC for optimal Treasury holdings. To get an additional unit of Treasury bonds today, the agent needs to give up some housing services (credit goods) consumption today, while his gain is the principal plus interest rate tomorrow, which can be transformed into housing services consumption tomorrow. So, the trade-off is between waiting to consume housing services tomorrow and paying house (rental) price appreciation versus the interest received. Note that this does not apply to cash goods consumption, which is constrained by the amount of cash.

In our cash-in-advance economy, housing, i.e., the credit good, expenditure is closely linked to the money supply chosen by the central bank while the cash good spending is sometimes disconnected. The price level of the cash good is constrained by the initial money supply, and is linked to the money supply chosen by the central bank as well as the housing rental price, but only during a recession (i.e., when the CIAC is not binding). When the economy is booming (the CIAC is binding), the money supply chosen by the central bank does not have an impact on the price level today while it does affect the interest rate and the housing rental price. The housing price is highly correlated with the money supply, and our theory provides an explanation for that and suggests that the optimal monetary policy should pay close attention to asset markets, in particular, the housing market.

Proposition 5: In any period, there exists *some output level* with non-zero measure, which after being observed by the central bank, causes the central bank to optimally choose a money supply such that the economy booms (i.e., CIAC is binding) regardless of the initial money supply.

Proof: See Appendix 1. ■

The above proposition says that, regardless of the initial money supply, the central bank will optimally conduct expansionary monetary policy, creating a boom, but only for certain output levels. Intuitively, ex-ante, money and Treasuries have the same expected value for the representative agent in equilibrium (otherwise only one of them will be held by the agent in equilibrium). However, we know that if the economy is in recession (CIAC is not binding), Treasuries are more valuable as they pay interest, and on the other hand, if the economy is booming (CIAC is binding), cash is more valuable as there is a shortage of cash. If the economy is almost surely in recession, Treasuries will always be more valuable unless the interest rate is zero, but zero interest rate cannot be optimal as there will be a very large amount of MBS in the economy and cost of financial fragility is prohibitively high.

However, when the initial money supply is very high, the chance of getting into recession becomes substantial, as we describe in the following proposition.

Proposition 6 (Boom-Bust): Assume that output is smoothly distributed for any amount of collateral, and in particular, we assume $\underline{y}\mathrm{E}[1/y'] \leq \underline{\pi}$. When the initial money supply is high enough, then there exists some output level with non-zero measure, which after being observed by the central bank, causes the central bank to optimally choose a money supply that triggers a recession (i.e., CIAC is not binding).

Proof: See Appendix 1. ■

Intuitively, with a large initial money supply, *M*, when the output is low, the central bank has to choose a relatively low level of money supply *M*’ to drive down the price this period with CIAC not binding. Otherwise, the price level this period would be too high, and the central bank has to choose a very high *M*’ to keep the economy away from deflation, but this cannot be optimal as the risk of financial fragility is too high.

4. Conclusion

The transformation of the global economy over the last thirty years has dramatically increased the role of collateral. Bank loans, which were previously passively held on bank balance sheets as immobile collateral became mobile via securitization. In the credit boom preceding the crisis, it was mortgage-backed securities that grew enormously. A large portion of Treasuries are held abroad, so in the United States, the ratio of MBS to Treasuries rose dramatically, ending in a financial crisis. Privately-produced collateral is not riskless, so when the ratio of MBS to Treasuries increases, financial fragility increases.

In a world where privately-produced collateral, MBS, is important, the central bank needs to respond to a decline in collateral quality to reduce financial fragility. The central bank undertakes this macroprudential role by incorporating the costs of financial fragility into its policy. In the setting here, it is the real value of collateral which is central to monetary policy. But, trading one kind of money for another complicates monetary policy.

We characterize the model by treating it as a dynamic game between the central bank, a large player, and the economic agents, a continuum of small players. The Perfect Public Equilibrium of the game resolves the issue of dynamic consistency, and we further refine the equilibrium to characterize the model in terms of recursive functions.

In such a dynamic game, the small agents rationally expect the central bank’s behavior, but they do not fully internalize the cost of financial fragility. The central bank is choosing the optimal money supply to balance the gain from higher output against the cost of financial fragility, and in equilibrium, the economy will experience deflation in recession, and we will observe a boom-bust pattern.

Appendix 1: Proofs

Proof of Lemma 1: With the multipliers $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, the Lagrange conditions for (2) are:

$$\begin{aligned} &v_c(c_2) - \lambda_1 p_2 - \lambda_2 p_2 = 0 \\ &\alpha v_c(nw_3 / p_2) / p_2 - \lambda_1 = 0 \\ &p_2 c_2 + w_3 \leq p_2 y_2 + nw_2 \\ &p_2 c_2 \leq m_2. \end{aligned}$$

It is easy to show that the budget constraint is binding, which implies $\lambda_1 > 0$. Eliminating λ_1 from the Lagrangean and substituting in the market clearing conditions, $c_2 = y_2$, $c_3 = nw_3 / p_2 = (w_3 - TB_2)/p_2 = M_2/p_2$, we have:

$$\begin{aligned} &v_c(y_2) - \alpha v_w(M_2 / p_2) - \lambda_2 p_2 = 0 \\ &p_2 y_2 \leq M_2. \end{aligned}$$

If the cash-in-advance condition is not binding, that is, $p_2 < M_2/y_2$, then $\lambda_2 = 0$, or, $v_c(y_2) = \alpha v_c(M_2 / p_2)$. With the assumptions on the utility function, we have

$$\begin{aligned} &v_c(y_2) = 1/ y_2 = \alpha v_c(M_2 / p_2) = \alpha p_2 / M_2 \\ &\Rightarrow p_2 = M_2 / \alpha y_2 > M_2 / y_2 \end{aligned}$$

which constitutes a contradiction.

Therefore, we must have the cash-in-advance constraint binding with $\lambda_2 > 0$, and we have $p_2 = M_2 / y_2$. ■

Proof of Lemma 2: With the multipliers $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, the Lagrange conditions for (2) are:

$$\begin{aligned} &v_c(c_2) - \lambda_1 p_2 - \lambda_2 p_2 = 0 \\ &\alpha v_c(nw_3 / p_2) / p_2 - \lambda_1 = 0 \\ &p_2 c_2 + w_3 \leq p_2 y_2 + nw_2 \\ &p_2 c_2 \leq m_2. \end{aligned}$$

It is easy to show that the budget constraint is binding, which implies $\lambda_1 > 0$. Eliminating λ_1 from the Lagrangean and using $p_2 = M_2 / y_2$, when the cash-in-advance constraint is binding, we have:

$$\begin{aligned} &c_2 = m_2 / p_2 = m_2 y_2 / M_2 \\ &nw_3 = M_2 + (tb_2 - TB_2)(1 + r_1) \end{aligned}$$

If the following condition holds, then the cash-in-advance constraint is binding:

$$m_2 < nw_3 / \alpha \text{ with } nw_3 = M_2 + (tb_2 - TB_2)(1 + r_1).$$

In which case:

$$\begin{aligned} U(m_2, tb_2) &= \ln(c_2) + \alpha \ln(nw_3 / p_2) \\ &= \ln(m_2 f / M_2) + \alpha \ln(nw_3 f / M_2) \\ &= (1+\alpha)\ln(f) + \ln(m_2 / M_2) + \alpha \ln(nw_3 / M_2). \end{aligned}$$

When the cash-in-advance constraint is not binding, we have $\lambda_2 = 0$ and

$$\begin{aligned} c_2 &= nw_3 / \alpha p_2 = nw_3 y_2 / \alpha M_2 \\ nw_3 &= \frac{\alpha}{1+\alpha}\left[M_2 + m_2 + (tb_2 - TB_2)(1+r_1)\right] \end{aligned}$$

Using the condition in the lemma, we have:

$$\begin{aligned} U(m_2, tb_2) &= \ln(c_2) + \alpha \ln(nw_3 / p_2) \\ &= \ln(nw_3 f / \alpha M_2) + \alpha \ln(nw_3 f / M_2) \\ &= (1+\alpha)\ln(f) + (1+\alpha)\ln(nw_3 / M_2) - \ln(\alpha). \blacksquare \end{aligned}$$

Proof of Lemma 3: At date 1, with the multipliers $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$, the Lagrange conditions for (1) are:

$$\begin{aligned} & u_c(c_1, h) - \lambda_1 p_1 - \lambda_2 p_1 = 0 \\ & u_h(c_1, h) - \lambda_1 q = 0 \\ & \beta U_m - \lambda_1 = 0 \\ & \beta U_{tb} - \lambda_1 = 0 \\ & p_1 c_1 + qh + m_2 + tb_2 \leq p_1 y_1 + qH + nw_1 \\ & p_1 c_1 \leq m_1. \end{aligned}$$

Substituting in the market clearing conditions $m_2 = M_2$ and $tb_2 = TB_2$ into $U(m_2, tb_2)$, we have:

$$\begin{aligned} U_m(m_2 = M_2, tb_2 = TB_2) &= 1/M_2 \\ U_{tb}(m_2 = M_2, tb_2 = TB_2) &= \alpha(1+r_1)/M_2. \end{aligned}$$

For the Lagrange conditions, substitute in the market clearing conditions, $c_1 = y_1$, $h = H$, and we have:

$$\begin{aligned} & \lambda_1 = \beta / M_2, 1 + r_1 = 1/\alpha, q = M_2 / \beta H \\ & 1/y_1 - \lambda_1 p_1 - \lambda_2 p_1 = 0 \\ & p_1 y_1 \leq M_1. \end{aligned}$$

When the cash-in-advance constraint *is* binding, we have:

$$
\begin{aligned}
&p_1 = M_1 / y_1 \\
&1 / y_1 - \lambda_1 p_1 = \lambda_2 p_1 > 0 \Rightarrow M_2 > M_1 \beta.
\end{aligned}
$$

When the cash-in-advance constraint is *not* binding, we have:

$$
\begin{aligned}
&\lambda_2 = 0 \\
&p_1 = 1 / y_1 \lambda_1 = M_2 / \beta y_1 \\
&p_1 y_1 = M_2 / \beta < M_1 \Rightarrow M_2 < M_1 \beta.
\end{aligned} \qquad \blacksquare
$$

Proof of Proposition 1: The output can be written as:

$$
y_2 = \begin{cases} \dfrac{A[\beta NW + (\delta\alpha/(1-\alpha) - \beta)M_2]y_1}{M_2} & \text{if } M_2 < M_1\beta \\[2ex] \dfrac{A[\beta NW + (\delta\alpha/(1-\alpha) - \beta)M_2]y_1}{\beta M_1} & \text{if } M_2 > M_1\beta. \end{cases}
$$

The result is immediate. ■

Proof of Lemma 4: The inflation rate can be written as:

$$
\pi = p_2 / p_1 = \begin{cases} \dfrac{\beta y_1}{A\dfrac{[\beta NW + (\delta\alpha/(1-\alpha) - \beta)M_2]y_1}{M_2}} & \text{if } M_2 < M_1\beta \\[4ex] \dfrac{M_2 y_1}{M_1 A\dfrac{[\beta NW + (\delta\alpha/(1-\alpha) - \beta)M_2]y_1}{\beta M_1}} & \text{if } M_2 > M_1\beta \end{cases}
$$

$$
= \frac{\beta M_2}{A[\beta NW + (\delta\alpha/(1-\alpha) - \beta)M_2]}.
$$

The result is immediate. ■

Proof of Lemma 5: It is easy to check that marginal utility, $\dfrac{(1+\alpha)(\delta\alpha/(1-\alpha)-\beta)y_1}{M_1 L(M_2)}$, is decreasing with M_2, and the marginal cost from financial fragility, $\dfrac{\gamma}{NW - M_2}$, is increasing with M_2. M_ψ satisfies $\dfrac{(1+\alpha)(\delta\alpha/(1-\alpha)-\beta)y_1}{M_1 L(M_\psi)} = \dfrac{\gamma}{NW - M_\psi}$, and thus yields the highest welfare for any $M_2 > M_1\beta$. The assumption $M_\psi > M_\pi$ guarantees that the deflation bound is not hit.■

Proof of Proposition 2: With the assumption that $M_\psi > M_\pi$, we know that the lower bound on the money supply is M_π. Proposition 2 tells us that when $M_\psi > M_\pi > M_1\beta$, the optimal money supply is M_ψ as M_ψ yields the highest welfare for any $M_2 > M_1\beta$. When $M_\psi > M_1\beta > M_\pi$, we know that in the region with $M_2 < M_1\beta$, the optimal money supply is M_π, as both the output and the risk of financial fragility increase as M_2 decreases, while in the region $M_2 > M_1\beta$, the optimal money supply is M_ψ, by Proposition 2, therefore the optimal money supply is M_π or M_ψ, whichever gives the highest value of social welfare. When $M_1\beta > M_\psi > M_\pi$, we know in the region $M_2 > M_1\beta$, social welfare is decreasing with M_2, while in the region $M_2 < M_1\beta$, social welfare is decreasing with M_2; therefore, the lower bound on money supply, M_π, yields the highest welfare, and is the optimal money supply. ■

Proof of Corollary 2: In Proposition 2, we can see that the statement is true when $M_\psi > M_\pi > M_1\beta$ or $M_1\beta > M_\psi > M_\pi$, so we only need to show the case with $M_\psi > M_1\beta > M_\pi$. We first prove that when M_1 satisfies $M_\psi > M_1\beta > M_\pi$, $V(M_\pi|M_1)/W(M_\psi|M_1)$ is increasing with M_1. To see that, we have:

$$V(M_\pi \mid M_1) = \ln y_1 + \ln H + \beta(1+\alpha)\ln\frac{A[\beta NW + (\delta\alpha/(1-\alpha)-\beta)M_\pi]y_1}{M_\pi} - \gamma \ln\frac{NW}{NW - M_\pi},$$

$$V(M_\psi \mid M_1) = \ln y_1 + \ln H + \beta(1+\alpha)\ln\frac{A[\beta NW + (\delta\alpha/(1-\alpha)-\beta)M_\psi]y_1}{\beta M_1} - \gamma \ln\frac{NW}{NW - M_\psi}.$$

The result is immediate as $V(M_\pi|M_1)$ is independent of M_1 while $V(M_\psi|M_1)$ is decreasing with M_1. We can check that, when $M_1 = M_\pi/\beta$, we have $V(M_\pi|M_1) < V(M_\psi|M_1)$; when $M_1 = M_\psi/\beta$, we have $V(M_\pi|M_1) > V(M_\psi|M_1)$. Therefore, there exists some $M_1^* \in (M_\pi/\beta, M_\psi/\beta)$ such that $V(M_\pi|M_1^*) = V(M_\psi|M_1^*)$.■

Proof of Proposition 3: We first prove that X_{t+1} and Z_{t+1} are bounded from above. We can write X_{t+1} and Z_{t+1} as follows:

$$\begin{aligned}
X_{t+1} &= E_B[u_c(y_{t+1},H)/p_{t+1}]\Pr(B) + E_{NB}[u_c(y_{t+1},H)c^{NB}_{w,t+1} + u_h(y_{t+1},H)h^{NB}_{w,t+1}]\Pr(NB) \\
Z_{t+1} &= E_B[u_h(y_{t+1},H)(1+r_t)/q_{t+1}]\Pr(B) \\
&\quad + E_{NB}[u_c(y_{t+1},H)c^{NB}_{w,t+1}(1+r_t) + u_h(y_{t+1},H)h^{NB}_{w,t+1}(1+r_t)]\Pr(NB)
\end{aligned}$$

which, as shown earlier, can be further simplified as

$$\begin{aligned}
X_{t+1} &= E[u_c(y_{t+1},H)/p_{t+1}] \\
Z_{t+1} &= E[u_h(y_{t+1},H)(1+r_t)/q_{t+1}]
\end{aligned}$$

When the cash-in-advance constraint is binding at time t, we know

$$\begin{aligned}
p_t &= M_t/y_t < NW/\underline{y} \\
p_t &= M_t/y_t > \underline{M}/\overline{y}.
\end{aligned}$$

Therefore, the cash goods price is bounded from above and below away from zero when the cash-in-advance constraint is binding. Moreover, when the cash-in-advance constraint is binding at time *t*+1, we also have

$$u_c(y_{t+1}, H) / p_{t+1} > u_h(y_{t+1}, H) / q_{t+1}.$$

When the cash-in-advance constraint is *not* binding at time *t*+1, we have the following results

$$u_c(y_{t+1}, H) / p_{t+1} = u_h(y_{t+1}, H) / q_{t+1}$$

It is easy to see that the cash goods price at time *t*+1 is always bounded from above as $p_{t+1}y_{t+1} < M_{t+1} < NW$, and this implies that the housing services price is also bounded from above with bounded u_c and u_h. Next, we will show that cash goods price and housing services price are also bounded from below with the lower limit strictly bigger than zero (this is critical to prove X_t and Z_t are bounded from above) when the cash-in-advance constraint is *not* binding.

Assume the housing services price is *not* bounded away from zero, then there exists a sequence of outputs $\{y_n\}_{n=1,\infty}$, and $\lim_{n\to\infty}q_n = 0$. As output is defined on a compact set, $[\underline{y}, \bar{y}]$, there exists an output level y^* as the limit of a subsequence of $\{y_n\}_{n=1,\infty}$, such that $q^* = 0$. However, if $q^* = 0$, the demand for housing services would be infinity, which cannot be sustained in equilibrium as the supply of housing services is a fixed value, *H*. Therefore, we conclude that, when the cash-in-advance constraint is *not* binding, the housing services price is bounded from below away from zero.

Moreover, taking the derivative with respect to m_{t+1} and tb_{t+1} for (4), we have

$$u_h(y_t, H) / q_t = \beta X_{t+1}$$
$$u_h(y_t, H) / q_t = \beta Z_{t+1}$$

Therefore, we can further conclude that X_{t+1} and Z_{t+1} are bounded from above.

Next, we prove the equivalence between the outcome of the auxiliary dynamic competitive equilibrium and that of the static one-period equilibrium. The necessity part is trivial. We now prove sufficiency. Assume $\{c_t^*, h_t^*, m_{t+1}^*, tb_{t+1}^*\} \in CE(m_t, tb_t, X_{t+1}, Z_{t+1})$. Then, a necessary condition for $\{c_t^*, h_t^*, m_{t+1}^*, tb_{t+1}^*\}$ to be the equilibrium outcome is:

$$u_c(c_t^*, h_t^*) / p_t \geq u_h(c_t^*, h_t^*) / q_t = \beta X_{t+1} = \beta Z_{t+1},$$

where the first inequality becomes a strict inequality when the cash-in-advance constraint is binding. By the concavity of *u*(*c*,*h*), for any $\{c_t, h_t, m_{t+1}, tb_{t+1}\} \neq \{c_t^*, h_t^*, m_{t+1}^*, tb_{t+1}^*\}$ we have

$$
\begin{aligned}
u(c_t,h_t) &\le u(c_t^*,h_t^*) + \begin{pmatrix} u_c(c_t^*,h_t^*) & u_h(c_t^*,h_t^*) \end{pmatrix} \begin{pmatrix} c_t - c_t^* \\ h_t - h_t^* \end{pmatrix} \\
&= u(c_t^*,h_t^*) + \begin{pmatrix} u_c(c_t^*,h_t^*)/p_t & u_h(c_t^*,h_t^*)/q_t \end{pmatrix} \begin{pmatrix} p_t(c_t - c_t^*) \\ q_t(h_t - h_t^*) \end{pmatrix} \\
&\le u(c_t^*,h_t^*) + \begin{pmatrix} u_h(c_t^*,h_t^*)/q_t & u_h(c_t^*,h_t^*)/q_t \end{pmatrix} \begin{pmatrix} p_t(c_t - c_t^*) \\ q_t(h_t - h_t^*) \end{pmatrix} \\
&= u(c_t^*,h_t^*) + \beta X_{t+1}\left(p_t(c_t - c_t^*) + q_t(h_t - h_t^*)\right) \\
&= u(c_t^*,h_t^*) + \beta X_{t+1}\left\{\left[(m_t - m_t^*) + (tb_t - tb_t^*)(1+r_{t-1})\right] - \left[(m_{t+1} - m_{t+1}^*) + (tb_{t+1} - tb_{t+1}^*)\right]\right\},
\end{aligned}
$$

where for the second inequality we use the fact that when the cash-in-advance constraint is binding, $u_c(c_t^*,h_t^*)/p_t \ge u_h(c_t^*,h_t^*)/q_t$ and $c_t \le c_t^*$.

We also have

$$
\begin{aligned}
&\beta X_{t+1}\left[(m_t - m_t^*) + (tb_t - tb_t^*)(1+r_{t-1})\right] \\
&= \begin{pmatrix} u_h(c_t^*,h_t^*)/q_t & u_h(c_t^*,h_t^*)(1+r_{t-1})/q \end{pmatrix} \begin{pmatrix} m_t - m_t^* \\ tb_t - tb_t^* \end{pmatrix} \\
&\le \begin{pmatrix} X_t & Z_t \end{pmatrix} \begin{pmatrix} m_t - m_t^* \\ tb_t - tb_t^* \end{pmatrix},
\end{aligned}
$$

where for the inequality we use the fact that when the cash-in-advance constraint is binding, $u_c(c_t^*,h_t^*)/p_t \ge u_h(c_t^*,h_t^*)/q_t$ and $m_t \ge m_t^*$.

Therefore, we have

$$
u(c_t,h_t) \le u(c_t^*,h_t^*) + \begin{pmatrix} X_t & Z_t \end{pmatrix} \begin{pmatrix} m_t - m_t^* \\ tb_t - tb_t^* \end{pmatrix} - \beta\begin{pmatrix} X_{t+1} & Z_{t+1} \end{pmatrix} \begin{pmatrix} m_{t+1} - m_{t+1}^* \\ tb_{t+1} - tb_{t+1}^* \end{pmatrix}.
$$

Adding these inequalities for different *t* together:

$$
\begin{aligned}
&\lim_{T\to\infty} E\left[\sum_{t=1}^{T} \beta^{t-1}[u(c_t,h_t) - u(c_t^*,h_t^*)]\right] \\
&\le \lim_{T\to\infty} E\left[\sum_{t=1}^{T} \beta^{t-1}\left[\begin{pmatrix} X_t & Z_t \end{pmatrix} \begin{pmatrix} m_t - m_t^* \\ tb_t - tb_t^* \end{pmatrix} - \beta\begin{pmatrix} X_{t+1} & Z_{t+1} \end{pmatrix} \begin{pmatrix} m_{t+1} - m_{t+1}^* \\ tb_{t+1} - tb_{t+1}^* \end{pmatrix}\right]\right] \\
&= \lim_{T\to\infty} E\left[-\beta^T\begin{pmatrix} X_{T+1} & Z_{T+1} \end{pmatrix} \begin{pmatrix} m_{T+1} - m_{T+1}^* \\ tb_{T+1} - tb_{T+1}^* \end{pmatrix}\right] = 0.
\end{aligned}
$$

Since X_{t+1} and Z_{t+1} are bounded from above. Therefore, $\{c_t^*, h_t^*, m_{t+1}^*, tb_{t+1}^*\}$ is optimal. ■

Proof of Proposition 4: Given the pricing functions $p(M, M', y)$ and $q(M, M', y)$, for any value function, $U(M, y)$, the central bank chooses next period money supply, M', to maximize the representative's utility.

Let C_U be the set of continuous functions from $[0, NW]\times[\underline{y}, \bar{y}]$ to R, and let ρ be the sup-norm defined on C_U, that is $\rho(V_1, V_2) = \sup\{V_1(M, y) - V_2(M, y)\}$. Given continuous $p(M, M', y)$, $q(M, M', y)$ and $V(M, y)$, define a mapping T_U: $C_U \rightarrow C_U$ as follows:

$$T_1(V)(M,y) = \max_{M'}\{u(y,H) - \psi(MBS/TB') + \beta E_{y'}[V(TB',y')]\}$$
$$s.t. \quad y' = f\left(\frac{TB' + \delta MBS}{p}, \varepsilon'\right)$$

We can show that $T_1(V)$ is a contraction mapping. According to Berge's Maximum Theorem, we know that there exists a fixed point, V, which is continuous and satisfies $T_1(V) = V$. Moreover, the corresponding solution $M'(M, y)$ is compact-valued upper hemi-continuous correspondences ($\underline{M}'(M, y)$ is not empty-valued as the objective function is continuous and the choice sets are compact by the Extreme Value Theorem. We know that a compact-valued upper hemi-continuous correspondence contains a continuous function,[6] and we can pick such a continuous function from $\underline{M}'(M, y)$, defined as $M'(M, y)$. Therefore, we have a map from continuous pricing functions ($p(M, M', y)$, $q(M, M', y)$) to $M'(M, y)$, which is also continuous.

Given continuous $M'(M, y)$, with $M'' = M'(M', y')$, define:

$$T_q(p) = u_h(y,H)/\tilde{q}(TB,TB',y) = \beta E[u_c(y',H)/p(TB',TB'',y')]$$
$$T_p(p) = u_c(y,H)/\tilde{p}(TB,TB',y)$$
$$= \begin{cases} u_c(y/H)y/(NW-TB) & \text{if } u_c(y/H)y/(NW-TB) > u_h(y,H)/\tilde{q}(TB,TB',y) \\ T_q(p) = u_h(y,H)/\tilde{q}(TB,TB',y) & \text{otherwise} \end{cases}$$

Basically, T_p maps p to $\tilde{p}$ through T_q (and $\tilde{q}$) using the first order condition with respect to m' for the representative agent's optimization problem.

Let C_p be the set of continuous function from $[0, NW]\times[\underline{y}, \bar{y}]$ to R, and let ρ be the sup-norm define on C_p, we can show $T_p(.)$ is a contraction mapping, and there exists a continuous

[6] To see this, we know that a compact-valued upper hemi-continuous correspondence (Ω: $X \rightarrow Y$) has the following property: for every sequence $\{x_n\} \rightarrow x$ and every sequence $\{y_n\}$ such that $y_n \in \Omega(x_n)$ for all n, there exists a convergent subsequence of $\{y_n\}$ whose limit point y is in $\Omega(x)$. See, for example, Stokey et al. (1989).

function $p(M, M', y)$ such that, $u_c(y, H)/\ p(M, M', y) = T_p(p)$. Thus, we establish a mapping from $M'(M, y)$ to $p(M, M', y)$, from which we can derive $q(M, M', y)$ from T_q.

With C being the set of continuous pricing functions from $[0, NW]^2 \times [\underline{y}, \bar{y}]$ to R^2, so far we have established a mapping from ($p(M, M', y)$, $q(M, M', y)$)$\in C$ to ($p(M, M', y)$, $q(M, M', y)$)$\in C$, which we call T. We can show that C is a non-empty weakly compact convex subset of a Banach space (because the set of continuous functions defined on a compact set with the sup-norm is compact, convex, and complete), and T is a continuous mapping as it is the product of two continuous mappings. By the Brouwer-Schauder-Tychonoff Fixed Point Theorem, we know there exists a fixed point $\{p(M, M', y), q(M, M', y)\}\in C([0, NW]^2 \times [\underline{y}, \bar{y}])$ such that $(p, q)\in T(p, q)$. [7] ■

Proof of Lemma 6: We know that: $1/qH = \beta E[Z(M', y')] = \beta E[(1+r)/q'H]$, which implies:

$$\frac{1}{\beta(1+r)} = E[q/q'] = 1 - E[(q'-q)/q'] \approx 1 - E[(q'-q)/q] = 1 - g_q.$$

Taking logs we have $r \approx g_q + (1-\beta)$. ■

Proof of Proposition 5: Suppose for almost all realizations of output y', the economy is in recession next period, i.e. the cash-in-advance constraint is not binding Then we have:

$$\begin{aligned} 1/qH &= \beta E[X(M', y')] = \beta E[1/q'H] \\ 1/qH &= \beta E[Z(M', y')] = \beta E[(1+r)/q'H], \end{aligned}$$

which implies $r = 0$. However, $r = 0$ implies MBS = infinity, which yields an infinitely high cost of financial fragility and cannot be an equilibrium outcome. This is never optimal, so the central bank adopts an expansionary monetary policy. ■

Proof of Proposition 6: Given that the initial money supply M large enough (close to NW), assume for any output y, the optimal monetary policy is such that the economy is booming (CIAC is binding) this period. For the lower bound of output level, $\underline{y}$, the price level this period will be $p = M/\underline{y}$. We know that the price level for next period must satisfy $p' \leq M'/y'$ by CIAC, and this implies $M' > M$ as we know $E[p']/p \geq \underline{\pi}$ but $yE[1/y'] \leq \underline{\pi}$. When M approaches

[7] See, for example, Aliprantis and Border (1999).

the upper limit, *NW*, we must have M’ approaching *NW*, which implies $\rho = MBS/TB'$ is approaching infinity and the cost of financial fragility goes to infinity. Therefore, the money supply *M*’ such that $M' > M$ and the economy is always booming cannot be the optimal monetary policy for the central bank when the output is low. ■

Appendix 2: Factorization of Perfect Public Equilibrium

In this appendix, we next demonstrate the recursive factorization of the PPE, following Abreu et al. (1990). Given the initial money supply in the economy $M \in [0, NW]$, define $\Gamma(M)$ to be the set of values which the representative agent can obtain and the marginal values of cash and Treasuries for the representative agent from a symmetric sequential equilibrium:

$$\Gamma(M) = (V(M), X(M), Z(M)) = \{(V(\sigma), X(\sigma), Z(\sigma)) \mid \sigma \text{ is a PPE for the economy } \Phi(M)\}.$$

We demonstrate the factorization of $\Gamma(M)$ by relating it to an arbitrary value correspondence, $W: [0, NW] \to R^3$, which is compact and convex.

Definition A1 A vector $\zeta = (M'(y), V'(y), X'(y), Z'(y))$ is said to be consistent with respect to W at the initial money supply, M, if the following conditions are satisfied:

1. Generation: $(V'(y), X'(y), Z'(y)) \in W(M)$;

2. Bounded monetary policy: $M'(y) \in [0, NW]$;

3. Representative agent's optimality: $(c(y), h(y), m'(y), tb'(y)) \in CE(M, TB, X'(y), Z'(y))$ with price $p(y)$ and $q(y)$, where $c(y) = y$, $h(y) = H$, $m'(y) = M'(y) = NW - TB'(y)$, $tb'(y) = TB'(y)$.

In the above definition, price $p(y)$ and $q(y)$ can be derived from the market clearance condition in the competitive economy $CE(M, TB, X'(y), Z'(y))$.

With Definition A1, we can define the worst payoff for the government for its choice of $M'(y)$. For each y, let:

$$\begin{aligned} &\underline{V}(M, y, M') \equiv \min_{(U', X', Z')} \left[u(c,h) - \psi(M,y) + \beta V'\right] \\ &s.t. \quad \zeta = (M', V', X', Z') \text{ is consistent with respect to } W \text{ at } M \\ &\qquad\quad \text{with } c \text{ and } h \text{ being the corresponding consumptions} \end{aligned}$$

We can use $\underline{V}(M, y, M')$ to define the punishment value for the central bank when it deviates from the equilibrium path.

Define

$$\overline{\underline{V}}(M, y) \equiv \max_{M' \in [0, NW]} \underline{V}(M, y, M')$$

Therefore, $\overline{\underline{V}}(M, y)$ is the best alternative value to the central bank when it deviates from the equilibrium path.

Definition A2 A vector ζ is said to be admissible with respect to *W* at the initial money supply, *M*, if ζ is consistent with *W* at *M* and the following two conditions are satisfied:

$$u(c(y),h(y))-\psi(M,y)+\beta V'(y)\geq \underline{V}(M,y) \text{ for each } y,$$

with *c*(*y*) and *h*(*y*) being the consumptions associated with ζ.

Admissibility adds the central bank's incentive constraint to the requirements for consistency. When the central bank chooses an unexpected money supply, the representative agent's beliefs are updated in the subsequent subgame so as to yield the worst possible payoff for the central bank. This is without loss of generality. If an equilibrium is sustainable with another type of punishment upon deviation, then it must also be sustainable with the worst punishment upon deviation. This is the same concept as what Chari and Kehoe (1990) call a "sustainable equilibrium."

Admissible $\zeta = (M'(y), V'(y), X'(y), Z'(y))$ gives the value which the representative agent can obtain and the marginal values of cash and Treasuries for the representative agent:

$$\begin{aligned}\widetilde{V}(M,\zeta) &\equiv E[u(c(y),h(y))-\psi(M)+\beta V'(y)]\\ \widetilde{X}(M,\zeta) &\equiv E[u_c(c(y),h(y))/p(y)]\\ \widetilde{Z}(M,\zeta) &\equiv E[u_h(c(y),h(y))(1+r_0)/q(y)]\end{aligned}$$

Let $\Xi(M,\zeta)\equiv(\widetilde{V}(M,\zeta),\widetilde{X}(M,\zeta),\widetilde{Z}(M,\zeta))$, and for a value correspondence *W*, define:

$$B(W)(M)=\{\Xi(M,\zeta)\mid \zeta \text{ is admissible with respect to } W \text{ at } M\}$$

Proposition A1 If *W* has a compact graph, then *B*(*W*) has a compact graph. Γ(*M*) is the largest value correspondence *W* such that *W* = *B*(*W*)(*M*).

Proof: First, we can show that *B*(*W*) has a bounded graph. It is easy to see that $V(M)\in[(u(\underline{y},H)-\overline{\psi})/\beta,(u(\overline{y},H)-\underline{\psi})/\beta]$. Also we have shown in the proof of Proposition 5 that *X* and *Z* are bounded from below and above.

Second, we can show that *B*(*W*) has a closed graph. Let $\{w_n, tb_n\}$ be a sequence in the graph of *B*(*W*) which converges to a point (*w*, *tb*). We need to show that (*w*, α) is also in the graph of *B*(*W*). By the definition of *B*(*W*), there exists a sequence of vectors $\zeta_n = (M'_n(y), V'_n(y), X'_n(y), Z'_n(y))$ admissible with respect to *W* at M_n, and $w_n = \Xi(M_n, \zeta_n)$. Because the space of admissible stage strategies and value functions are bounded, we may assume this sequence converges to some limit point $\zeta = (M'(y), V'(y), X'(y), Z'(y))$, where the convergence of functions $\{M'_n(y)\}$, $\{V'_n(y)\}$, $\{X'_n(y)\}$ and $\{Z'_n(y)\}$ is almost everywhere.

To show the almost-everywhere convergence of $\{M'_n(y)\}$, $\{V'_n(y)\}$, $\{X'_n(y)\}$ and $\{Z'_n(y)\}$, we proceed as follows. First, all these functions are bounded on Y, which is itself bounded. Therefore, these functions are in L_p-space. It is easy to find a Cauchy subsequence for each of them, say $\{g_n\}$, and we know that there exists a function g in L_p-space with a L_p-norm, s.t. $\lim_{n\to\infty} \| g_n - g \|_p \to 0$. Second, $\lim_{n\to\infty} \| g_n - g \|_p \to 0$ in L_p-space implies that there exists a subsequence $\{k_n\}$ of $\{g_n\}$, such that $\{k_n\}$ converges to g almost everywhere. This guarantees that g is a feasible function, that is, it is constrained by the same bounds as that for the original function sequence, $\{M'_n(y)\}$, $\{V'_n(y)\}$, $\{X'_n(y)\}$ or $\{Z'_n(y)\}$.

By continuity of Ξ, we know that $\Xi(M, \zeta) = w$, and ζ is admissible with respect to W at α. Therefore, (w, α) is in the graph of $B(W)$. Therefore, $B(W)$ has a compact graph.

Following Phelan and Stacchetti (2001), by definition of PPE, we can show that $\Gamma(M) = B(\Gamma)(M)$ for all M, and $\Gamma(M)$ is the largest value correspondence W such that $W = B(W)(M)$. For each M, define $W_\infty(M) = B(B(...(B(W)))(M) = B_\infty(W)(M)$, and with the compactness property of $B(.)$, we can show $W_\infty(M) = \Gamma(M)$. ■

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