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Abstract: We study a decentralised and uncoordinated market where heterogeneous self-interested firms and workers meet directly and randomly in pursuit of higher payoff over time. Each firm hires several workers and each worker has preferences over firms and salaries, taking at most one job. Neither firms nor workers possess perfect knowledge of the market. At any time any firm and any group of workers can form a new coalition if doing so makes no member of the coalition worse off and at least one member strictly better off. In this process, the firm may recruit workers from other firms and dismiss some of its own workers, the deserted firms and fired workers can be worse off. This process is called the coalition improvement. We establish that starting from an arbitrary market state of a matching between firms and workers with a system of salaries, a decentralised random dynamic market process where each possible coalition improvement occurs with a positive probability converges with probability one to a competitive equilibrium, provided that an equilibrium exists. This theorem is built upon a crucial mathematical result which shows the existence of a finite sequence of successive coalition improvements from an arbitrary market state to equilibrium. Our results also have meaningful policy implications.

Keywords: Decentralised market, labour market, random dynamic process, competitive equilibrium, spontaneous market process, indivisibility.

JEL classification: C62, C68, D02, D44, J02.

"Every individual endeavors to employ his capital so that its produce may be of greatest value. He generally neither intends to promote the public interest, nor knows how much he is promoting it. He intends only his own security, only his own gain. And he is in this led by an invisible hand to promote an end which was no part of his intention. By pursuing his own interest he frequently promotes that of society more effectually than when he really intends to promote it." Quoted from Adam Smith (1776), *The Wealth of Nations*

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1 Introduction

One of the central issues of economic research is to study market processes by which equilibrium prices or wages can be formed. The basic idea of market processes can be traced back at least to Smith (1776), who coined the famous term, the Invisible Hand, to describe the self-regulating nature of an uncoordinated market. Walras (1874) suggested a price adjustment process known as the tâtonnement process. In it, a fictitious auctioneer announces a price for one good, collecting all the demands for the good, adjusting the price by the law of demand and supply until an equilibrium on this single market is reached. The same procedure applies to the remaining goods successively one by one. Obviously, this procedure is very restrictive. A major improvement was made by Samuelson (1941, 1948) who proposed a simultaneous tâtonnement process. Arrow and Hahn (1948), and Arrow, Block and Hahn (1949) proved that Samuelson's process converges globally to an equilibrium provided that all goods are perfectly divisible and substitutable. Scarf (1960) showed by examples that this process, however, does not work if goods are complementary. Later, Scarf (1973) invented a path-breaking process that always leads to an equilibrium in any plausible market with divisible goods. More recently, efficient market processes such as auctions and job matching have been developed to deal with economies with significant indivisibilities; see Crawford and Knoer (1981), Kelso and Crawford (1982), Gul and Stacchetti (2000), Milgrom (2000, 2004), Ausubel and Milgrom (2002), Perry and Reny (2005), Ausubel (2004, 2006), and Sun and Yang (2009, 2014) among others. A common feature of these processes is that all economic activities are coordinated by an auctioneer or a clearing house in a deterministic and orderly manner.

This paper aims to study a decentralised, random, and dynamic market process in which heterogeneous self-interested firms and workers meet directly and randomly in pursuit of higher payoffs over time. This process intends to mimic and reflect decentralised and uncoordinated decision making in real labour markets. In the market under consideration, each firm hires as many workers as it wishes, having a revenue value for each group of workers. Each worker has preferences over firms and salaries but works for at most one firm. Neither firms nor workers are assumed to have perfect knowledge of the market. Each agent (firm or worker) makes her own decision independently and freely. At any time any firm and any group of workers can form a new coalition by dividing the joint payoff among all its members in a specific way if doing so makes no member of the coalition worse off and at least one member strictly better off. In this process, the firm may dismiss some of its own workers and recruit workers from other firms to be called deserted firms, and each deserted firm will continue to hire its remaining workers. This process is called the coalition improvement and only assumed to occur with a positive probability conditional on the current state. The assumption on occurrence probability is intended to capture the essential features of the labour market that are pervasive uncertainty about market opportunities as pointed out by Kelso and Crawford (1982, p.1483). Also decentralised and uncoordinated decision making in complex real economic environments inevitably brings in uncertainty or randomness; see Roth and Vande Vate (1990, p. 1475). Furthermore, the assumption is a natural requirement that although information about the market is dispersed and imperfect, it flows freely enough so that all market participants are sufficiently well informed and can therefore respond to newly arrived opportunities. It could be viewed as a degree of market transparency.

The above decentralised, random and dynamic process will be simply called a spontaneous process which is the result of human action but not of conscious human design such as auction or matching design. In Hayek (1988), it is described as a spontaneous order. This process exhibits several features which are widely observed in many real life uncoordinated markets including labour markets. First, the deserted firms and fired workers are generally worse off and thus the total welfare in the process need not be monotonic. Second, a worker may sequentially work for several employers because a latter employer offers a better salary than a previous employer does or the worker may have been fired previously; and conversely, a same firm hires different workers over time for the same positions as workers who come later may either work more efficiently or demand lower salaries. In addition, it is not uncommon to see that a worker eventually returns to her previous employer but with a different contract. Third, if a firm loses its employee(s) to competing firms, it will continue to hire its remaining employees without changing their contracts at least for a short period of time. This is a common practice in real business. For instance, if a star professor moves from university A to university B, the former will not change at least temporally its contracts with the remaining faculty members. Fourth, the process allows a firm in debt up to a certain level to immediately declare bankrupt by firing all its employees or continue to run by hiring and firing and reorganising, as in real business. Fifth, the process is random and dynamic, and can be chaotic and sometimes temporally cyclical as firms and workers meet directly and randomly, haggling for better deals, and coalitions can be formed probabilistically and hastily and can also dissolve instantly whenever better opportunities arise.

An intriguing, natural and fundamental question arises here: will such decentralised, chaotic, random and dynamic processes eventually settle the market in an equilibrium state in which a system of competitive salaries exists and simultaneously meets the needs of both firms and workers? This paper attempts to resolve this question in the affirmative. Our main result establishes that starting from an arbitrary market state of a matching between firms and workers with a system of salaries, the above general decentralised, dynamic, and random market process where each possible coalition improvement conditional on the current state occurs with a positive probability converges in finite time with probability one to a competitive equilibrium of the market consisting of an efficient matching between firms and workers and a scheme of supporting salaries (see Theorem 1 and Corollary 1), resulting in a Pareto optimal outcome. This result is surprisingly general in the sense that it holds true for any market environments as long as there exists a competitive equilibrium with an integral or rational vector of equilibrium salaries or prices. The existence of such equilibrium prices is a natural and practical assumption, because any transaction in real business can only happen in rational or integer number of monetary units. A number of sufficient conditions are known to ensure the existence of such equilibrium; see Kelso and Crawford (1982), Bikhchandani and Mamer (1997), Ma (1998), Gul and Stacchetti (1999, 2000), Milgrom (2000, 2004), Ausubel (2004, 2006), Sun and Yang (2006, 2008, 2009), Milgrom and Strulovici (2009). Among them, the Gross Substitutes condition introduced by Kelso and Crawford (1982) has been widely used and requires every firm to view all workers as substitutes, including the classical assignment model by Koopmans and Beckmann (1957), Shapley and Shubik (1971), Crawford and Knoer (1981), Demange, Gale and Sotomayor (1986) as a special case.[4] A crucial step toward establishing the major result of the current paper is to prove that the decentralised, random and dynamic process does not get stuck in cycles endlessly. To this end, we develop a novel thought experiment technique to show the existence of a finite sequence of successive coalition improvements from an arbitrary initial market state to a competitive equilibrium of the market (Theorem 2).

To the best of our knowledge, the results we obtain in this paper are the first most general results for decentralised, random and dynamic competitive market processes under a mild condition that the market has an integral or rational vector of equilibrium salaries or prices. We believe our results provide a theoretical foundation for validating Adam Smith's Invisible Hand in complex real economic environments involving uncertainty, indivisibility and imperfect information and offers a plausible explanation for a large class of decentralised, random and dynamic competitive market processes. An unintended consequence of the current study is that our results seem to have some interesting policy implication: in general, free markets can work wonders as Hayek (1944) and Friedman (1962) passionately advocated, almost surely resulting in socially efficient outcomes even in a chaotic, random and imperfect information environment, and in particular the price system can marvellously aggregate and communicate information "in a system in which the knowledge of the relevant facts is dispersed among many people" as Hayek (1945, pp. 525-527) had deeply believed. A caveat is that free markets cannot unconditionally function well, and

[4]Such models are also called unit-demand models and assume that every consumer demands at most one item (see also Shapley and Scarf 1974) or every person needs only one opposite sex partner (see Gale and Shapley 1962).

in order for them to work well and speedily, the government should promote and improve market transparency and provide some coordination.

The current work is closely related to several previous studies. In a pioneering study Kelso and Crawford (1982) introduced a general job matching market where each firm can hire several workers and each worker is employed by at most one firm. They developed a salary adjustment process that converges to an equilibrium provided that every firm treats all workers as substitutes. Although they stress that pervasive uncertainty is an essential feature of the labour market, they do not deal with uncertainty and their process is a deterministic market process. Chen, Fujishige and Yang (2010) examined a decentralised random process for the assignment market as studied by Koopmans and Beckmann (1957) and Shapley and Shubik (1971). In a seminal article Roth and Vande Vate (1990) reexamined the marriage matching model of Gale and Shapley (1962) in which each man tries to marry his favorite woman and vice versa, and established a decentralised random process for the model. Kojima and Ünver (2008) generalised the marriage model in an important way to allow for instance each college to admit many students and each student to attend several colleges. They investigated a decentralised random process for a pairwise stable matching outcome.

A crucial and well-recognised difference between the matching models as studied by Gale and Shapley (1962), Roth and Sotomayor (1990), Roth and Vande Vate (1990), and Kojima and Ünver (2008), and the competitive markets as studied by Koopmans and Beckmann (1957), Arrow and Hahn (1971), Shapley and Shubik (1971), Shapley and Scarf (1974), Kelso and Crawford (1982), and ourselves among others is that the matching models do not involve prices or a medium of exchange nor have a system of competitive prices to support a stable matching outcome, which is the often used notion of solution to matching models and generally weaker than the concept of competitive equilibrium; see for instance Quinzii (1984). Feldman (1974) and Green (1974) studied deterministic decentralised processes for certain subclasses of non-transferable utility games. Their approaches do not apply to the labour market or matching models where indivisibility is involved.

The paper is organised as follows. Section 2 presents the general framework and basic concepts. Section 3 contains the main results. Section 4 introduces the thought experimental procedure for proving the crucial mathematical result (Theorem 2). Section 5 examines the particular but very interesting case of Gross Substitutes. Section 6 concludes.

2 The Model and Basic Concepts

Consider a general labour market with finitely many heterogeneous firms and workers. Formally, let F be the set of m firms and W the set of n workers, respectively. We assume that each firm can hire as many workers as it wishes but each worker can work for at most one firm. Each firm $j \in F$ has a nondecreasing integer-valued revenue function $R^j : 2^W \to \mathbb{Z}$ with $R^j(\emptyset) = 0$. Namely, when firm j hires a group $B \subseteq W$ of workers, it has a revenue of $R^i(B)$ in units of money thus being an integer value. Given a scheme of salaries $s^j = (s_i^j \mid i \in W)$ for firm $j \in F$, firm j's net profits are given by $\pi_j(B, s^j) = R^j(B) - \sum_{i \in B} s_i^j$. Each worker $i \in W$ has quasi-linear utility in money and has an integer minimum wage requirement $w_i^j \geq 0$ for being willing to work at firm $j \in F$. Because of the minimum wage requirement for the same salary, worker i may prefer to be hired by firm j rather than by firm k. The integer value assumption of R^j and w_i^j is quite natural and standard, as for example we cannot specify a monetary payoff more closely than to its nearest penny.[5] The information about R^j and w_i^j can be private, as explained in the next section. We use $(F, W, (R^j \mid j \in F), (w_i^j \mid i \in W, j \in F))$ ((F, W, R^j, w_i^j), in short) to represent this economy. In addition, for any $F' \subseteq F$ and $W' \subseteq W$, let (F', W', R^j, w_i^j) be the economy only consisting of firms in F' and workers in W'. In the sequel a worker or firm may be simply called an agent.

A matching μ in the labour market is a correspondence such that for all $i \in W$, either $\mu(i) = i$ or $\mu(i) \in F$, for all $j \in F$, $\mu(j) \subseteq W$, and for all $i \in W$ and $j \in F$, $\mu(i) = j$ if and only if $i \in \mu(j)$. At matching μ, for any worker $i \in W$, if $\mu(i) \in F$, then $\mu(i)$ represents the firm to which worker i is assigned. If $\mu(i) = i$, then worker i is not assigned to any firm and we will say that such worker i is *unemployed* or *self-matched*. For any firm $j \in F$, $\mu(j)$ stands for the set of workers hired by firm j. If $\mu(j)$ is empty, then firm j does not employ any worker.

A salary scheme system $S = (s^j \mid j \in F)$ consists of every firm j's salary scheme s^j. A *state or allocation* of the market consists of a salary scheme system $S = (s^j \mid j \in F)$ and a matching μ. At allocation (μ, S), if $\mu(i) = j \in F$ for any worker $i \in W$, then worker i works at firm j and receives salary s_i^j; if $\mu(i) = i$, then worker i does not work for any firm and receives no salary, and firm j hires the group $\mu(j)$ of workers and pays the total amount $s^j(\mu(j)) = \sum_{i \in \mu(j)} s_i^j$ of salary. An allocation (μ, S) induces a payoff vector $u \in \mathbb{R}^{F \cup W}$ such that for every worker $i \in W$, $u_i = s_i^{\mu(i)} - w_i^{\mu(i)}$ when $\mu(i) \in F$, and $u_i = 0$ when $\mu(i) = i$, and for every firm $j \in F$, $u_j = \pi_j(\mu(j), s^j)$. In this way, the state (μ, S) can be alternatively written as (μ, u). Observe that at every state (μ, u) we have $u_j + \sum_{i \in \mu(j)} u_i = R^j(\mu(j)) - \sum_{i \in \mu(j)} w_i^j$ for every firm $j \in F$ and $u_k = 0$ for every $k \in W$ with $\mu(k) = k$.

A state (μ, u) is *individually rational* if no agent is worse than she stands alone, i.e., $u_k \geq 0$ for every $k \in F \cup W$. A nonempty group $B \subseteq F \cup W$ of workers and firms is called

[5]See e.g., Demange, Gale and Sotomayor (1986), Roth and Sotomayor (1990), Ausubel (2006), and Sun and Yang (2009).

a coalition. Following Kelso and Crawford (1982, p. 1487), we say that a coalition B is *essential* if it contains either only one worker or only one firm with any number of workers. Note that we make this definition slightly more general than theirs by including the case of either a single firm or a single worker to cover individual rationality. In the following any coalition means an essential coalition. Sometimes it is convenient to use (j, B) to express a coalition in order to distinguish the firm and workers, where $j \in F$ or $j = \emptyset$, and $B \subseteq W$.

A state (μ, u) is *weakly blocked by a coalition* $B \subseteq F \cup W$ if there exists one firm $j \in B$ with a payoff vector $r \in \mathbb{R}^B$ such that

$$r_k \geq u_k \text{ for every } k \in B \text{ and} \tag{1}$$

$$\sum_{k \in B} r_k = R^j(B \setminus \{j\}) - \sum_{k \in B \setminus \{j\}} w_k^j, \tag{2}$$

with at least one strict inequality for (1), or if the coalition B contains only one worker i with $0 > u_i$. Notice that if B contains only a firm j, then $r_j = 0$. The definition says that it would be better off for at least one member in B and none in B worse off if firm j hires only workers $i \in B$, and every worker $i \in B$ works for firm j, or it would be strictly better for the single firm not to hire any worker or for the single worker i not to work at firm $\mu(i)$. B is called a *weakly blocking coalition*. A *strongly blocking coalition* is defined in the same manner as a weakly blocking coalition, except that (1) is now strengthened as a strict inequality for every member in B.

A state is a *core allocation* if it is not strongly blocked by any coalition. Clearly, a state is a core allocation if and only if it is individually rational and is not strongly blocked by any firm with at least one worker. A state is a *strict core allocation* or *a competitive equilibrium* if it is not weakly blocked by any coalition. It is well known from Kelso and Crawford (1982, p. 1487) that the set of competitive equilibria in this market coincides with the set of strict core allocations.

Definition 1 *Let B be a blocking coalition against the state (μ, u). A new state (μ', u') is said to be* a coalition improvement of the state (μ, u) through B, *where the new state is constructed as follows:*

(1) *if there exists one firm $j \in B$ with the associated payoff vector $r \in \mathbb{R}^B$, let $\mu'(i) = j$ and $u'_i = r_i$ for every worker $i \in B$, $\mu'(j) = B \setminus \{j\}$ and $u'_j = r_j$, $\mu'(i) = i$ and $u'_i = 0$ for every worker $i \in \mu(j) \setminus B$, $\mu'(i) = \mu(i)$ and $u'_i = u_i$ for every worker $i \in W \setminus (B \cup \mu(j))$, $\mu'(k) = \mu(k) \setminus B$ and $u'_k = R^k(\mu(k) \setminus B) - \sum_{i \in \mu(k) \setminus B}(w_i^k + u_i)$ for every firm $k \in F \setminus \{j\}$; or*

(2) *if the condition B contains only one worker i, let $\mu'(i) = i$ and $u'_i = 0$, $\mu'(k) = \mu(k)$ and $u'_k = u_k$ for every worker $k \in W \setminus \{i\}$, $\mu'(\mu(i)) = \mu(\mu(i)) \setminus \{i\}$ and $u'_{\mu(i)} =$*

$R^{\mu(i)}(\mu(\mu(i)) \setminus \{i\}) - \sum_{k \in \mu(\mu(i)) \setminus \{i\}}(w_k^{\mu(i)} + u_k)$, $\mu'(k) = \mu(k)$ *and* $u'_k = u_k$ *for every firm* $k \in F \setminus \{\mu(i)\}$.

By definition, at the new state (μ', u'), firm j will hire all workers $i \in B$ and share the revenue according to the given specification $r \in \mathbb{R}^B$, whereas workers in $\mu(j)$ not in B will be fired by firm j and unemployed and get a payoff of zero, all other workers outside $B \cup \mu(j)$ will maintain the status quo of (μ, u). Observe that every firm $k \in F \setminus \{j\}$ will continue to hire those workers who are not in the blocking coalition B and who were hired by firm k at (μ, u), and that firm k's payoff may be negatively affected if some worker from firm k is hired by firm j, because firm k will keep the same contract with each of its remaining workers as in the state (μ, u). With respect to a coalition improvement of a state through the blocking coalition B, we also distinguish *weak coalition improvement* from *strong coalition improvement*, depending on whether the associated blocking coalition B is weak or strong.

Weak coalition improvements imitate real life business practices. When a firm and a group of workers find an opportunity to form a weakly blocking coalition, the firm and hired workers are better off but the deserted firms and dismissed workers are usually worse off. If an employee leaves a firm for a better offer from another firm, the abandoned firm usually will not immediately change contracts for its remaining employees and needs time to adapt to the new situation. It is fairly common that such a firm will continue to run for a period of time even if it may be in debt. If a worker is fired by a firm, she needs time to find a new job. In such processes, it is not unusual to observe that as in reality, a worker may jump from one position to another and may eventually return to her previous employer, but with different contracts. Weak coalition improvements also allow a firm in debt to immediately adjudicate it bankrupt by firing all its employees or continue to run by hiring and firing and reorganising, as in real life business.

It should be noticed that no firm $j \in F$ will offer any worker $i \in W$ a salary more than its maximum revenue $R^j(W)$. In the sequel, any market state (μ, u) will be understood as a state in which this property always holds true. Such a state is said to be *feasible*.

Observe that we define all concepts such as blocking coalition and competitive equilibrium based on real numbers. However, most real life market processes work only on rational or integral salaries or prices. The following lemma shows that, assuming integrality of revenue functions and minimum salary requirements, an integral state (μ, u) is a competitive equilibrium within the domain of real payoffs if (and only if) it is not weakly blocked by any coalition with integral payoffs. It should be noticed that this result holds true without requiring any other extra conditions.

Lemma 1 *Let* R^j *and* w_i^j *for all* $i \in W$ *and* $j \in F$ *be integral. If a state* (μ, u) *with* $u \in \mathbb{Z}^{F \cup W}$ *is not weakly blocked by any coalition* B *with integral payoffs* $v_i \in \mathbb{Z}$ *for all*

$i \in B$, then it cannot be blocked by any coalition T with real payoffs $u'_i \in \mathbb{R}$ for all $i \in T$. Consequently, (μ, u) must be a competitive equilibrium.

Proof. Suppose to the contrary that an integral state (μ, u) with $u \in \mathbb{Z}^{F \cup W}$ which is not blocked by any group of firm j and workers B with integral payoffs $v_j, v_i \in \mathbb{Z}$ for all $i \in B$, is blocked by a group of firm k and workers T with real payoffs $u'_k, u'_i \in \mathbb{R}$ for all $i \in T$. Because the coalition $T \cup \{k\}$ blocks (μ, u), then we have

$$u'_k + \sum_{i \in T} u'_i = R^k(T) - \sum_{i \in T} w^k_i, \tag{3}$$

$$u'_k \geq u_k \text{ and } u'_i \geq u_i \text{ for all } i \in T \tag{4}$$

with at least one strict inequality. Let $K = \{i \in T \mid u'_i > u_i \text{ and } u'_i \notin \mathbb{Z}\} \cup \{k \mid u'_k > u_k \text{ and } u'_k \notin \mathbb{Z}\}$. If K is empty, we have a contradiction. If K is not empty, it follows from (3) and the integer number $R^k(T) - \sum_{i \in T} w^k_i$ that K contains at least two elements. Take any element $i^* \in K$. Then let $\bar{u}_i = u_i (\in \mathbb{Z})$ for every $i \in K$ with $i \neq i^*$ and $\bar{u}_i = u_i (\in \mathbb{Z})$ for every $i \in (T \cup \{k\}) \setminus K$, and $\bar{u}_{i^*} = R^k(T) - \sum_{i \in T} w^k_i - \sum_{i \in (T \cup \{k\}) \setminus \{i^*\}} u_i (> u_{i^*})$. $\bar{u}_{i^*}$ is an integer. Because u_i for all $i \in F \cup W$, R^j for all $j \in F$, and w^j_i for all $j \in F$ and $i \in W$ are integers, we have the coalition $T \cup \{k\}$ with integer payoffs $\bar{u}_k$ and $\bar{u}_i$ for all $i \in T$ that blocks (μ, u), yielding a contradiction. The case of a singleton coalition is easy to verify. This completes the proof. □

For convenience, a state (μ, u) with an integral payoff vector $u \in \mathbb{Z}^{F \cup W}$ or equivalently integral salaries or prices will be called *an integral state*. We are particularly interested in integral states because transactions in real world business can happen only in integral or rational number of monetary units. The above lemma shows that it is sufficient to concentrate on integral states. There are several major sufficient conditions guaranteeing the existence of an integral competitive equilibrium. The most well-known of these conditions is the Gross Substitutes condition of Kelso and Crawford (1982) which will be introduced shortly.

Given a salary scheme $s^j \in \mathbb{R}^W$, let $D^j(s^j)$ be the set of solutions to

$$\max_{T \subseteq W} \pi_j(T, s^j)$$

$D^j(s^j)$ is the collection of those groups of workers which give the firm the highest profit at the offered salaries s^j.

Definition 2 *Firm j satisfies the Gross Substitutes condition if for every salary scheme $s^j \leq t^j$ and $A \in D^j(s^j)$, there exists $C \in D^j(t^j)$ such that $\{i \mid i \in A \text{ and } s^j_i = t^j_i\} \subseteq C$.*

This condition states that if a firm j hires a group A of workers at salaries s^j and if the salaries are now increased to the new levels t^j, the firm will still want to hire those workers in A whose salaries do not increase.

It is well known that this job matching market admits at least one competitive equilibrium and that the set of strict core allocations coincides with that of competitive equilibria (Kelso and Crawford 1982). In addition, as all valuations w_i^j and R^j are integers and every firm satisfies the Gross Substitutes condition, the labour market must have at least one strict core allocation with an integral payoff vector $u \in \mathbb{Z}^{F\cup W}$ or an integral salary system $S = (s^1, s^2, \cdots, s^m) \in \mathbb{Z}^{W\times F}$; see Gul and Stacchetti (1999), Ausubel (2006), and Sun and Yang (2009). Notice that the celebrated assignment market of Koopmans and Beckmann (1957) and Shapley and Shubik (1971) automatically satisfies the Gross Substitutes condition and thus has an integral equilibrium. Another basic condition for the existence of integral equilibrium is the Gross Substitutes and Complements condition of Sun and Yang (2006, 2009) which generalises the Gross Substitutes condition by permitting a typical pattern of complementarity.

3 Decentralised Random Competitive Processes

In this section we address the central issue whether a spontaneous, decentralised, random and uncoordinated market process can settle the market in a competitive equilibrium or not. Suppose that the market starts at time 0 with an arbitrary integral state. It is plausible to assume that information about the market is dispersed among all the market participants and no single agent or organisation commands complete knowledge of the market. For instance, each firm j possesses private information about its own revenue function R^j and each worker i knows her own minimum wage requirement w_i^j privately. Because firms and workers are self-interested, any individual or group of agents will be willing to grasp any opportunity to improve their wellbeing by forming a new coalition within which the firm may fire some of its workers and hire some workers from other firms, and some workers may abandon their employers. The formation of the new coalition is a weak coalition improvement against the current state. Because agents are not assumed to have perfect information of the market and decentralised decision making in real life environments inevitably involves a certain level of uncertainty or randomness, such coalition improvements cannot be expected to occur with absolute certainty but with a positive probability. Because real life transactions take place only in integral or rational number of monetary units, it suffices to work with only integral weak coalition improvements. Obviously, this random, decentralised and spontaneous process will continue to move from one disequilibrium state to another until a competitive equilibrium is reached. A natural and fundamental question arises here: will such a random, decentralised and spontaneous market process converge to a competitive equilibrium eventually? The following theorem gives an affirmative answer by showing that this general process will converge probabilistically to a competitive equilibrium in finite time, provided that at any point in time, every weak coalition improvement of the current market state arises with a positive probability. The assumption of a positive probability for every weak coalition improvement also implies that although information about the market is imperfect and totally dispersed among all the agents involved, job-related information flows smoothly enough so that agents can grasp newly arrived opportunities in the market at least with a positive probability.

As it will be clear in the following proof of Theorem 1, the requirement of the positive probability being no less than any fixed small number $\varepsilon > 0$ is really minimal. This probability could be viewed as a measure of market transparency. The magnitude of this positive number ε will not affect the convergence of the decentralised random competitive process but it does have an impact on the convergence speed. In general, the bigger ε is, the faster the process will be.

Theorem 1 *Assume that the labour market* (F, W, R^j, w_i^j) *has a competitive equilibrium with an integral equilibrium payoff vector. Starting with an arbitrary integral market state, any random and decentralised process, where only every integral weak coalition improvement occurs with a positive probability, converges to a competitive equilibrium in finite time with probability one.*

Corollary 1 *Assume that every firm in the market* (F, W, R^j, w_i^j) *satisfies the Gross Substitutes condition. Then starting with an arbitrary integral market state, any random and decentralised process, where only every integral weak coalition improvement occurs with a positive probability, converges to a competitive equilibrium in finite time with probability one.*

The proof of this result relies on the following crucial mathematical theorem, which establishes a link between any integral initial market state and an integral competitive equilibrium through only a finite sequence of successive weak coalition improvements. The distinguishing feature of finite successive weak coalition improvements is essential to capture the decentralised nature of the random market process. Any other path which does not exhibit this feature but may still connect the initial market state with a competitive equilibrium will not achieve the goal. It is also worth pointing out that the proof of Theorem 1 depends critically on the statement of Theorem 2 but not on its proof technique or procedure.

Theorem 2 *Assume that the labour market* (F, W, R^j, w_i^j) *has a competitive equilibrium with an integral equilibrium payoff vector. Starting with an arbitrary integral market state, there exists a finite number of successive weak coalition improvements leading to a competitive equilibrium.*

We now discuss how to establish Theorem 1 via Theorem 2. As pointed out in the previous section, it is sufficient and also natural to confine ourselves to integral market states. We can further concentrate on feasible integral market states, because no firm is willing to pay any worker more than its maximum revenue. Let $\mathcal{A}(F, W)$ denote the set of all feasible integral market states. Observe that $\mathcal{A}(F, W)$ is nonempty and finite, as the number of workers and firms is finite, the number of matchings is finite, any value of R^j is finite, and every feasible payoff vector is integral and bounded.

Suppose that the market starts with an arbitrary initial market state in $\mathcal{A}(F, W)$ at time $t = 0$, and runs every day $t = 1, 2, \cdots$. Consider a general decentralised random market process in which every time-dependent transition probability from a disequilibrium state in $\mathcal{A}(F, W)$ at any time t to another state in $\mathcal{A}(F, W)$ at time $t + 1$ is no less than a fixed (but sufficiently small) number $\varepsilon \in (0, 1)$, namely, every possible weak coalition improvement occurs with a positive probability. With only two classes of states (equilibrium and disequilibrium), it follows that starting from any state (μ, u) in $\mathcal{A}(F, W)$, the process either terminates in equilibrium state and remains in equilibrium afterwards, or continues to move from one disequilibrium state to another disequilibrium state in $\mathcal{A}(F, W)$, as the random process by construction always arrives at a state in $\mathcal{A}(F, W)$. Suppose that the random process does not converge to an equilibrium state with probability one in the limit. This implies that at some point, after reaching a disequilibrium state in $\mathcal{A}(F, W)$, the random process oscillates among a (finite) set of disequilibrium states in $\mathcal{A}(F, W)$ indefinitely. Since each possible weak coalition improvement is chosen with a probability no less than ε at each point of time, there is then some state (μ', u') in $\mathcal{A}(F, W)$ from which no finite path of weak coalition improvements toward equilibrium exists, no matter how the associated weak coalition improvements are chosen, yielding a contradiction to Theorem 2. This completes the proof of Theorem 1.

4 The Construction of a Desired Path to Equilibrium

We will prove Theorem 2 in this section. The key point is to construct a finite sequence of successive weak coalition improvements linking an arbitrary initial integral market state with a competitive equilibrium. This sequence is *our desired path* and any other path which does not generate successive weak coalition improvements but still leads to a competitive equilibrium will not help to establish the theorem.

Because the labour market (F, W, R^j, w_i^j) is assumed to have a competitive equilibrium with integral equilibrium payoffs, we can take any such competitive equilibrium (μ^*, u^*). We call (μ^*, u^*) the *reference equilibrium point.* The idea of using a reference point is a conventional and powerful thought experiment method and can avoid many practical issues and has been used in theoretical physics. Biró et al. (2012) use this idea to the unit-demand models such as assignment market and partnership formation problems. It should be emphasized here that our method of constructing a desired path is very general in the sense that it works for any market as long as the market has an equilibrium. The method serves the purpose of proving Theorem 2 but is not a practical economic adjustment process.

Given a state (μ, u), an agent $i \in F \cup W$ is *underpaid* (*overpaid*) at (μ, u) with respect to the reference point (μ^*, u^*) if $u_i \leq u_i^*$ ($u_i > u_i^*$). For any $U \subseteq W$ and $j \in F$ define $u(U) = \sum_{i \in U} u_i$ and $w^j(U) = \sum_{i \in U} w_i^j$. For convenience we also use π_j to stand for the payoff u_j of firm $j \in F$ in a state (μ, u).

We will describe a general procedure that starting from any initial market state (μ, u) generates a finite number of weak coalition improvements leading to a competitive equilibrium. We first give a subroutine UPDATE which will be repeatedly used in the procedure.

The subroutine UPDATE tells us how to adjust a state (μ, u) weakly blocked by the coalition (j, U) to a new state (μ', u'). It specifies one type of weak coalition improvement.

$$\mathsf{UPDATE}(\mu, u, j, U)$$

(a) Start from an integral market state (μ, u) weakly blocked by (j, U). Go to Step (b).

(b) Let $F' = \{k \in F \setminus \{j\} \mid \mu(k) \cap U \neq \emptyset\}$ and $F^* = \{k \in F \setminus \{j\} \mid \mu(k) \cap U = \emptyset\}$. If $j = \emptyset$, go to Step (e). If $j \in F$, make workers in $\mu(j) \setminus U$ unemployed and their payoffs at zero. Let $\mu'(i) = i$ and $u_i' = 0$ for every $i \in \mu(j) \setminus U$. Firm j hires all workers in U. Let $\mu'(j) = U$. Go to Step (c).

(c) If there exists an overpaid worker $i \in U$, then increase one such u_i' so that

$$u_i' + \sum_{l \in U \setminus \{i\}} u_l = R^j(U) - \pi_j - w^j(U),$$

let $u_l' = u_l$ for every $l \in U \setminus \{i\}$ and $\pi_j' = \pi_j$, and then go to Step (e). Otherwise, go to Step (d).

(d) There exists no overpaid worker in U. Increase all underpaid workers' u_i' as much as possible in such a way that $u_i' \leq u_i^*$ with $u_i' \in \mathbb{Z}$ and $u'(U) \leq R^j(U) - \pi_j - w^j(U)$. If $u'(U) = R^j(U) - \pi_j - w^j(U)$, let $\pi_j' = \pi_j$ and go to Step (e). Otherwise let $\pi_j' = R^j(U) - u'(U) - w^j(U)$ and go to Step (e).

(e) For each $k \in F'$, let $U_k = \mu(k) \setminus U$. Every firm $k \in F'$ continues to hire all remaining workers in U_k under the same contract as in (μ, u). Let $u'_i = u_i$ for every $i \in U_k$, $\mu'(k) = U_k$ and $\pi'_k = R^k(U_k) - u(U_k) - w^k(U_k)$. For each $k \in F^*$, let $\pi'_k = \pi_k$ and $u'_i = u_i$ for every $i \in \mu(k)$. Go to Step (f).

(f) Set $(\mu, u) = (\mu', u')$ and get a new integral market state (μ, u).

The process consisting of Steps (a), (b), (c) and (d) will be called **Reshuffle**(μ, u, j, U) while the process consisting of only Step (e) will be called **Retention**(μ, u, j, U).

If a state (μ, u) is weakly blocked by a coalition (j, U) with $j \in F$ and $U \subseteq W$, by definition we have

$$R^j(U) - w^j(U) > \pi_j + u(U).$$

Since (μ^*, u^*) is a competitive equilibrium, it follows that

$$\pi^*_j + u^*(U) \geq R^j(U) - w^j(U) > \pi_j + u(U).$$

This implies that UPDATE can be executed.

Lemma 2 *After* UPDATE*, if firm j's payoff π'_j increases, we have $u'_i = u^*_i$ for all $i \in U$ and also $\pi'_j \leq \pi^*_j$.*

Proof. Note that π'_j gets increased only if Step (d) is executed, i.e., $\pi'_j = R^j(U) - u'(U) - w^j(U)$ and $u'_i = u_i$ for all $i \in U$. Since $\pi'_j + u^*(U) = \pi'_j + u'(U) = R^j(U) - w^j(U) \leq \pi^*_j + u^*(U)$, we have $\pi'_j \leq \pi^*_j$. □

Note that if U contains at least one overpaid worker, after the execution of UPDATE firm j's payoff remains the same as in the state (μ, u).

We are now ready to describe the procedure which, starting from an arbitrary integral market state, will generate a finite sequence of successive coalition improvements leading to a competitive equilibrium.

The Procedure for a Desired Path to Equilibrium

Step 0 Let (μ^*, u^*) be a reference equilibrium of the market. Start with an arbitrary integral market state (μ, u). Go to Step 1.

Step 1 If there exists a weakly blocking coalition (j, U) with $j \in F$ and $U \subseteq \mu(j) \cup \mu^*(j)$ against the current state (μ, u), go to Step 2. Otherwise, go to Step 3.

Step 2 Find an optimal solution U^* to the following problem

$$\begin{array}{ll} \max & R^j(X) - u(X) - w^j(X) \\ \text{s.t.} & X \subseteq \mu(j) \cup \mu^*(j). \end{array}$$

Perform $\mathsf{UPDATE}(\mu, u, j, U^*)$ and get a new state (μ, u). Go to Step 1.

Step 3 If the state (μ, u) is weakly blocked by a coalition (j, U), perform $\mathsf{UPDATE}(\mu, u, j, U)$ by giving a new state (μ, u) and go to Step 1. Otherwise stop with the current state (μ, u), which is a competitive equilibrium.

For convenience, the process that the Procedure goes through **Step 1** and **Step 2** before moving into **Step 3** is called **Phase 1**, while the process that the Procedure goes through **Step 3** before returning to **Step 1** is called **Phase 2**.

The following observation is simple but crucial to the convergence of the Procedure and follows immediately from the construction of the UPDATE process.

Observation 1: In the entire Procedure, every underpaid worker will always remain underpaid. If an overpaid worker becomes underpaid, she will remain underpaid afterwards.

Lemma 3 *Let (μ, u) be the state of the market with which the* Procedure *goes to* **Step 3**. *Then for all $j \in F$, $\mu(j)$ is a maximizer of $R^j(X) - u(X) - w^j(X)$ in $X \subseteq \mu(j) \cup \mu^*(j)$ and there exists no subset X of the set $\mu(j) \cup \mu^*(j)$ such that (j, X) weakly blocks (μ, u).*

Proof. By **Step 2** and UPDATE, for every $j \in F$, $\mu(j)$ is a maximizer of $R^j(X) - u(X) - w^j(X)$ in $X \subseteq \mu(j) \cup \mu^*(j)$. Hence,

$$\pi_j = R^j(\mu(j)) - u(\mu(j)) - w^j(\mu(j)) \geq R^j(X) - u(X) - w^j(X), \text{for all } X \subseteq \mu(j) \cup \mu^*(j).$$

It follows that there exists no $X \subseteq \mu(j) \cup \mu^*(j)$ such that (j, X) weakly blocks (μ, u). □

Lemma 4 *Every worker $i \in \mu(j) \backslash (U^* \cup \mu^*(j))$ that leaves firm j in* **Step 2** *of the* Procedure *will never return to the firm afterward before the* Procedure *goes to* **Step 3**.

Proof. Each worker $i \in \mu(j) \setminus (U^* \cup \mu^*(j))$ before the UPDATE in **Step 2** becomes self-employed and disappears from $\mu(j) \cup \mu^*(j)$ for the new $\mu(j)$ after the UPDATE in **Step 2**. Since during the repetition of **Step 1** and **Step 2** sets $\mu(j) \cup \mu^*(j)$ for all $j \in F$ do not get enlarged, such a worker i remains to be self-employed before **Step 3** is invoked. □

Lemma 5 *Every worker $i \in (\cup_{k \in F \setminus \{j\}} \mu(k)) \cap U^*$ that moves to firm j in* **Step 2** *of the* Procedure *will never return to her previous firm afterward (but possibly becomes unemployed by being fired by firm j) before the* Procedure *goes to* **Step 3**.

Proof. Similarly as the proof of the previous lemma, it follows again from the definition of the UPDATE process. □

Observe that every worker $i \in (\mu(j) \cap \mu^*(j)) \setminus U^*$ gets $u_i = 0$ and this will remain the same afterward before the Procedure goes to **Step 3**. Also note that if the set $\mu(j) \cup \mu^*(j)$ remains the same after UPDATE, i.e., only some workers in $\mu(j) \cap \mu^*(j)$ leave firm j, there is no weakly blocking coallition (j, U) with $U \subseteq \mu(j) \cup \mu^*(j)$ after the update, which can be seen by arguments similar to the proof of Lemma 3.

Lemma 6 *Let (μ, u) be the allocation with which the* Procedure *goes to* **Step 3**. *Then it holds*

$$\pi_j^* + u^*(\mu^*(j)) = \pi_j^* + u^*(\mu(j)) = \pi_j + u(\mu^*(j)) = \pi_j + u(\mu(j)) \ \textit{for all } j \in F. \tag{5}$$

Proof. It follows from Lemma 3 that for every firm $j \in F$

$$\begin{aligned} \pi_j^* + u^*(\mu^*(j)) &= R^j(\mu^*(j)) - w^j(\mu^*(j)) \\ &\leq R^j(\mu(j)) - u(\mu(j)) - w^j(\mu(j)) + u(\mu^*(j)) \\ &= \pi_j + u(\mu^*(j)). \end{aligned} \tag{6}$$

On the other hand, since (μ^*, u^*) is a competitive equilibrium, we have for all $j \in F$

$$\begin{aligned} \pi_j^* + u^*(\mu(j)) &\geq R^j(\mu(j)) - w^j(\mu(j)) \\ &= \pi_j + u(\mu(j)). \end{aligned} \tag{7}$$

It follows from (6) and (7) that

$$\begin{aligned} \sum_{j \in F}(\pi_j + u(\mu(j))) &\leq \sum_{j \in F}(\pi_j^* + u^*(\mu(j))) \\ &\leq \sum_{j \in F}(\pi_j^* + u^*(\mu^*(j))) \\ &\leq \sum_{j \in F}(\pi_j + u(\mu^*(j))) \\ &\leq \sum_{j \in F}(\pi_j + u(\mu(j))). \end{aligned} \tag{8}$$

Hence every inequality in (6)–(8) hold with equality. This leads to

$$\pi_j^* + u^*(\mu^*(j)) = \pi_j^* + u^*(\mu(j)) = \pi_j + u(\mu^*(j)) = \pi_j + u(\mu(j)) \, \text{for all} \, j \in F.$$

□

The above proof and lemma also imply

Lemma 7 *Let* (μ, u) *be the allocation with which the* Procedure *goes to* **Step 3**. *It holds that*

$$u_i = 0 \text{ for all } i \in W \setminus \bigcup_{j\in F} \mu^*(j) \text{ and } u_i^* = 0 \text{ for all } i \in W \setminus \bigcup_{j\in F} \mu(j).$$

Now, we examine the behaviour of **Step 3** of the Procedure. Let (μ, u) be the state at the beginning of an execution of **Step 3**. Take any weakly blocking coalition (j, U) and define $F' = \{k \in F \mid U \cap \mu(k) \neq \emptyset\}$. Because of the definitions of (j, U) and (μ^*, u^*) we have

$$\pi_j + u(U) < R^j(U) - w^j(U) \leq \pi_j^* + u^*(U). \tag{9}$$

If $j \notin F'$, let $F' = F' \cup \{j\}$. Then from (5) we have

$$\begin{aligned} &\sum_{k\in F'} (\pi_k + u(\mu(k))) \\ &= \sum_{k\in F'\setminus\{j\}} (\pi_k + u(\mu(k) \setminus U)) + u(\mu(j) \setminus U) + (\pi_j + u(U)) \\ &= \sum_{k\in F'} (\pi_k^* + u^*(\mu(k))). \end{aligned} \tag{10}$$

It follows from (9) and (10) that for some firm $k \in F' \setminus \{j\}$

$$\pi_k + u(\mu(k) \setminus U) > \pi_k^* + u^*(\mu(k) \setminus U) \tag{11}$$

or

$$u(\mu(j) \setminus U) > u^*(\mu(j) \setminus U). \tag{12}$$

Case (I): (11) holds. It follows from (11) and the equilibrium (μ^*, u^*) that we have

$$\pi_k + u(\mu(k) \setminus U) > \pi_k^* + u^*(\mu(k) \setminus U) \geq R^k(\mu(k) \setminus U) - w^k(\mu(k) \setminus U). \tag{13}$$

Then by Step (e) of UPDATE we have

$$\pi_k' = R^k(\mu(k) \setminus U) - w^k(\mu(k) \setminus U) - u(\mu(k) \setminus U).$$

Hence π_k', the new π_k, strictly decreases compared with previous π_k in (13).

Case (II): (12) holds. Then there exists at least one overpaid worker in $\mu(j) \setminus U$. By Step (b) of UPDATE all workers $i \in \mu(j) \setminus U$ become unemployed and at least one overpaid worker in $\mu(j) \setminus U$ becomes underpaid.

We can now establish the following major mathematical result of this section and thus prove Theorem 2.

Theorem 3 *The* Procedure *generates a finite sequence of weak coalition improvements leading to an integral competitive equilibrium.*

Proof. Recall that the number of integral feasible market states is finite. Furthermore, every market state generated by the Procedure is an integral feasible state. If the Procedure does not produce a finite sequence of weak coalition improvements leading to a competitive equilibrium, it must yield a finite cycle. The Procedure would repeat the cycle forever. Without loss of generality we assume that at least one **Phase 1** is executed before the Procedure reaches the cycle.

Notice that Case (II) in **Phase 2** never occurs along the cycle, because there are at most n overpaid workers and if any overpaid worker becomes unemployed, it will remain underpaid forever by Observation 1. Thus only Case (I) may occur in **Phase 2** along the cycle. Then, since in Case (I) π_k strictly decreases, π_k should be increased to recover the loss along the cycle, which can only be done by Reshuffle(μ, u, k, U) in **Phase 1** when $U(= \mu(k)$ later) does not contain any overpaid workers. Note that each Retention in **Phase 1** makes the value of π_k less than or equal to that of π_k given at the end of the previous **Phase 1**, since at the end of **Phase 1** then obtained $\mu(k)$ for every $k \in F$ is a maximizer obtained in **Step 2**, so that removing some workers from $\mu(k)$ results in a lower revenue than that given at the end of the previous **Phase 1** for firm k, while Retention in **Phase 2** may only reduce π_k because of the same reason. Also note that after the Reshuffle(μ, u, k, U) we have $\pi_k \leq \pi_k^*$ due to Lemma 2 and comments right after that and then keep $\pi_k \leq \pi_k^*$ thereafter. On the other hand, if $\pi_k \leq \pi_k^*$, then because of (11) in Case (I) in **Phase 2** there must be at least one overpaid worker in $\mu(k) \setminus U$, where we update $\mu(k)$ by setting $\mu(k) = \mu(k) \setminus U$. Hence along the cycle there exists at least one overpaid worker i who becomes unemployed and thus remains underpaid thereafter forever by Observation 1. But this is impossible along the cycle, because there are only at most n overpaid workers. In other words, there will be no overpaid worker along the cycle, yielding a contradiction. Hence the Procedure terminates (in finitely many integral weak coalition improvements) with a final integral state (μ, u) that has no integral weak blocking, which is a competitive equilibrium due to Lemma 1. □

5 The Benchmark Case of Gross Substitutes

Weak coalition improvements cover all kinds of hiring and firing procedures and some of these procedures could be too general and too complicated to handle. However, under the Gross Substitutes condition it is possible to obtain the following much simpler, more intuitive and more well-behaved form of hiring and firing procedure.

A weakly blocking coalition (j, B) against a state (μ, u) is called a *basic weakly blocking coalition* if either $j = \emptyset$ or $j \in F$ implies that one of the following holds:

(1) $B = \mu(j) \cup \{k\}$ for $k \in W \setminus \mu(j)$;

(2) $B = (\mu(j) \cup \{l\}) \setminus \{k\}$ for some worker $k \in \mu(j)$ and some worker $l \in W \setminus \mu(j)$;

(3) $B = \mu(j) \setminus \{k\}$ for some worker $k \in \mu(j)$.

A weak coalition improvement (μ', u') of (μ, u) through (j, B) is called a *basic weak coalition improvement* if (j, B) is a basic weakly blocking coalition. With respect to (j, B), in Case (1), firm j hires a new worker, in Case (2), firm j simultaneously dismisses a worker and hires a new worker, and in Case (3), firm j fires a worker.

It is immediately clear that a basic weakly blocking coalition (j, B) against a state (μ, u) occurs if and only if either $j = \emptyset$ or $j \in F$ implies that one of the following occurs:

(1) For a firm $j \in F$ and a worker $k \in W \setminus \mu(j)$,

$$u_j + \sum_{i \in \mu(j)} (u_i + w_i^j) + u(k) + w_k^j < R^j(\mu(j) \cup \{k\}).$$

(2) For a firm $j \in F$, a worker $k \in W \setminus \mu(j)$, and a worker $l \in \mu(j)$,

$$u_j + \sum_{i \in \mu(j) \setminus \{l\}} (u_i + w_i^j) + u(k) + w_k^j < R^j((\mu(j) \cup \{k\}) \setminus \{l\}).$$

(3) For a firm $j \in F$ and a worker $k \in \mu(j)$,

$$u_j + \sum_{i \in \mu(j) \setminus \{k\}} (u_i + w_i^j) < R^j(\mu(j) \setminus \{k\}).$$

The following important characterisation is called the *Single Improvement* (SI) property and shown by Gul and Stacchetti (1999,2000) to be equivalent to the Gross Substitutes condition of Kelso and Crawford (1982).

Definition 3 *Firm j satisfies the Single Improvement property if for every salary scheme s^j and $A \notin D^j(s^j)$, there exists $B \in D^j(s^j)$ such that $R^j(B) - \sum_{i \in B} s_i^j > R^j(A) - \sum_{i \in A} s_i^j$, $|A \setminus B| \leq 1$ and $|B \setminus A| \leq 1$.*

Next we show that under the Gross Substitutes condition it is indeed sufficient to consider only basic weakly blocking coalitions.

Theorem 4 *Under the Gross Substitutes condition, if there is a weakly blocking coalition, there must be a basic weakly blocking coalition.*

Proof. Suppose we are given a weakly blocking coalition (j, B) for some firm $j \in F$ and worker group $B \subseteq W$. By definition, there exists a salary scheme t^j such that $t_i^j - w_i^j \geq s_i^{\mu(i)} - w_i^{\mu(i)}$ for every $i \in B$ and $\pi_j(B, t^j) \geq \pi_j(\mu(j), s^j)$ with at least one strict inequality.

Now define a new salary scheme $\tilde{t}^j \in \mathbb{R}^W$ by

$$\tilde{t}_i^j = s_i^{\mu(i)} - w_i^{\mu(i)} + w_i^j, \qquad \forall i \in W. \tag{14}$$

Observe that $\tilde{t}_i^j = s_i^{\mu(i)} - w_i^{\mu(i)} + w_i^j = s_i^j$ for every $i \in \mu(j)$, and $\tilde{t}_i^j = s_i^{\mu(i)} - w_i^{\mu(i)} + w_i^j \leq t_i^j - w_i^j + w_i^j = t_i^j$ for every $i \in B$. Then we have

$$\pi_j(B, \tilde{t}^j) \geq \pi_j(B, t^j) \geq \pi_j(\mu(j), s^j) = \pi_j(\mu(j), \tilde{t}^j). \tag{15}$$

It follows from the definition of (j, B) that at least one of the two inequalities in (15) is strict, i.e.,

$$\pi_j(B, \tilde{t}^j) > \pi_j(\mu(j), \tilde{t}^j). \tag{16}$$

Hence from (16) we have $\mu(j) \notin D^j(\tilde{t}^j)$. Now because of the GS condition it follows from the SI property one of the three cases must occur:

(4) there is a worker $k \in W \setminus \mu(j)$ such that

$$R^j(\mu(j) \cup \{k\}) - \sum_{i \in \mu(j) \cup \{k\}} \tilde{t}_i^j > R^j(\mu(j)) - \sum_{i \in \mu(j)} \tilde{t}_i^j.$$

(5) there are a worker $k \in W \setminus \mu(j)$ and a worker $l \in \mu(j)$ such that

$$R^j((\mu(j) \cup \{k\}) \setminus \{l\}) - \sum_{i \in (\mu(j) \cup \{k\}) \setminus \{l\}} \tilde{t}_i^j > R^j(\mu(j)) - \sum_{i \in \mu(j)} \tilde{t}_i^j.$$

(6) there is a worker $k \in \mu(j)$ such that

$$R^j(\mu(j) \setminus \{k\}) - \sum_{i \in \mu(j) \setminus \{k\}} \tilde{t}_i^j > R^j(\mu(j)) - \sum_{i \in \mu(j)} \tilde{t}_i^j.$$

Note that $\tilde{t}_i^j - w_i^j = s_i^{\mu(i)} - w_i^{\mu(i)} = u_i$ for all $i \in W$ due to (14). We first consider case (4). Since $u_i = \tilde{t}_i^j - w_i^j = s_i^j - w_i^j$ and $\tilde{t}_i^j = s_i^j$ for every $i \in \mu(j)$, and $u_j = R^j(\mu(j)) - \sum_{h \in \mu(j)} \tilde{t}_i^j = R^j(\mu(j)) - \sum_{h \in \mu(j)} s_i^j$, we have

$$R^j(\mu(j) \cup \{k\}) - \sum_{i \in \mu(j)} \tilde{t}_i^j - \tilde{t}_k^j > u_j$$

which can be written as

$$u_j + \sum_{i \in \mu(i)} (u_i + w_i^j) + u(k) + w_k^j < R^j(\mu(j) \cup \{k\})$$

This corresponds to case (1).

Similarly, using $u_j = R^j(\mu(j)) - \sum_{h\in\mu(j)} \tilde{t}_i^j = R^j(\mu(j)) - \sum_{h\in\mu(j)} s_i^j$, one can show that case (5)

$$R^j((\mu(j)\cup\{k\})\setminus\{l\}) - \sum_{i\in(\mu(j)\cup\{k\})\setminus\{l\}} \tilde{t}_i^j > R^j(\mu(j)) - \sum_{i\in\mu(j)} \tilde{t}_i^j,$$

implies case (2)

$$u_j + \sum_{i\in\mu(j)\setminus\{l\}} (u_i + w_i^j) + u(k) + w_k^j < R^j((\mu(j)\cup\{k\})\setminus\{l\}),$$

and that case (6) implies case (3). □

Under the Gross Substitutes condition we can establish the following major refinement of Theorem 2.

Theorem 5 *For the labour market (F, W, R^j, w_i^j) under the Gross Substitutes condition, there exists a finite number of basic weak coalition improvements from an arbitrary market state to a competitive equilibrium.*

By this result one can easily write down the corresponding refinement of Theorem 1 under the Gross Substitutes condition. The proof of the above theorem follows from the Procedure in the previous section, Theorem 3 and the next lemma.

Lemma 8 *In* **Step 2** *of the* Procedure*, there exists a finite sequence of basic weak coalition improvements from the current $\mu(j)$ to a maximizer U^* within $\mu(j)\cup\mu^*(j)$.*

Proof. Let $p_k = u_k + w_k^j$ for every $k \in \mu(j)\cup\mu^*(j)$. Consider the following problem

$$\begin{array}{ll} \max & R^j(X) - u(X) - w^j(X) = R^j(X) - \sum_{k\in X} p_k \\ \text{s.t.} & X \subseteq \mu(j)\cup\mu^*(j) \end{array}$$

Let $D^j(p)$ be the collection of optimal solutions to the problem. Since $D^j(p)$ satisfies the Gross Substitutes condition, it satisfies the Single Improvement property. Consequently, there exists a finite sequence of basic weak coalition improvements from the current $\mu(j)$ to a maximizer U^* within $\mu(j)\cup\mu^*(j)$. □

It is also possible to obtain a refined version of Theorems 1 and 2 under the Gross Substitutes and Complements condition of Sun and Yang (2006, 2009).

6 Conclusion

Economic processes are fundamental instruments by which markets are operated and equilibrium prices or salaries are generated. Such processes can be roughly classified into two major groups. One group comes out from deliberate human design, such as auctions, which have been widely used to sell mobile-phone licenses, electricity, treasure bills, mineral rights, keywords, pollution permits, and many other commodities and services involving a staggering value of hundreds of billions of dollars (see Krishna 2002, Klemperer 2004, and Milgrom 2004). In some sense, a conscious human design market process can be regarded as a visible hand. The other are spontaneous market processes which arise naturally from human economic action but are not designed by human being, and perhaps are literally the "true" invisible hand as conceived by Adam Smith. Uncoordinated decentralised markets such as labour markets are of this nature. While many important results have been obtained for the first type of market processes, we have far less understanding of the second. This paper shed light on the second type of market processes for a large class of general real decentralised markets.

In the paper we have analysed a general decentralised labour market where heterogeneous self-interested firms and workers meet directly and randomly in pursuit of higher payoff over time. Each firm hires as many workers as it wishes. Each worker has preferences over firms and salaries but works for at most one firm. Each economic agent makes her own decision independently and freely. The information of the market is dispersed among all separate market participants. In other words, information is imperfect and incomplete to every individual. At any time any firm and any group of workers can form a new coalition by dividing the joint payoff among all its members if doing so makes no member of the coalition worse off and at least one member strictly better off. In the process, the firm may fire some of its own workers and hire workers from other firms and each deserted firm will continue to hire its remaining workers. An important feature is that in the process the total welfare need not be monotonic, because every abandoned firm and dismissed worker could be worse off. As information is imperfect and decision-making is decentralised, it is natural to assume that this coalition improvement occurs only with a positive probability conditional on the current state. This random dynamic process captures several salient features of spontaneous processes that are widely observed in real decentralised markets. We have shown that starting with any initial market state, the decentralised random market process converges almost surely in finite time to a competitive equilibrium, thus resulting in an efficient allocation of resources. The result holds true for any competitive market as long as there exists an equilibrium with an integral vector of equilibrium salaries or prices. An important example for equilibrium existence is the well-known Gross Substitutes condition of Kelso and Crawford (1982).

We believe our results have laid a theoretical foundation for affirming Adam Smith's Invisible Hand in complex practical economic environments involving uncertainty, indivisibility and imperfect information and provided fresh insight into a large class of decentralised, random and dynamic competitive market processes. The current study offers also interesting and meaningful policy implications: Free markets can generally accomplish miracles in achieving efficient distribution of resources even in a chaotic, random and imperfect information environment. More specifically, the price system can play a vital role in efficiently communicating information "in a system in which the knowledge of the relevant facts is dispersed among many people, prices can act to coordinate the separate actions of different people in the same way as subjective values help the individual to coordinate the parts of his plan" as Hayek (1945, pp. 525-527) had so deeply believed. A word of caution is that in order for free markets to perform well, the government should improve market transparency and offer some coordination amongst market participants.

Our model is very general and natural in almost all respects but its zero search cost assumption. An important direction for future research is to relax this assumption. For instance, firms and workers do not and cannot always make contact with one another immediately. Firms are trying to find workers and workers are looking for jobs. This search process usually requires resources and time, thus creating frictions in the market; see e.g., Diamond (1971, 1981). How will such search frictions affect efficiency of the market and convergence of the spontaneous process?

We hope that the current study will prove to be useful in understanding fundamental issues concerning decentralised dynamic market processes in the complex real world.

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